LECTURE - 21

• Recall 1) Let $\gamma : [0,1] \longrightarrow M$ be a geodesic. Then $J \in \mathcal{J}(\gamma) \iff \exists$ a variation $\Gamma(s,t)$ of γ s.t. $\gamma_s(t) := \Gamma(s,t)$ is a geodesic $\forall$ s & $J = \partial_s \Gamma(0,t)$.

2) Jacobi equation $\quad D_t^2 J + R(J, \dot{\gamma})\dot{\gamma} = 0$.

3) Conjugate points For any $t \in (0,1]$, $\gamma(t)$ is called conjugate to p along γ if $\exp_p$ is singular at $v_t := tv$, where $v = \dot{\gamma}(0)$.

• CHARACTERIZATION OF CONJUGATE POINTS.

Prop 21.1: Let $\gamma : [0,1] \longrightarrow M$ be a geodesic s.t. $\gamma(0) = p$ & $\gamma(1) = q$. Then q is conjugate to p along $\gamma \iff$ $\exists$ a $J \in \mathcal{J}(\gamma)$ s.t. $J(p) = J(q) = 0$ ($\iff Null_\gamma \neq \emptyset$)
$J \not\equiv 0$.

Pf: Let $\gamma = \exp_p tv$. Then $q \in Conj_\gamma(p) \iff$ $\exists$ $w \in T_v T_p M \cong T_p M$ s.t. $(d\exp_p)_v(w) = 0$.

Also note that $J \in \mathcal{J}(\gamma)$ w/ $J(0) = 0$, $D_t J(0) = w$ $\iff$ $J(t) = (d\exp_p)_{tv}(tw)$. This is because

$$J(t) = \frac{d}{ds}\Big|_{s=0} \exp_p t(v + sw) = (d\exp_p)_{tv}(tw).$$

The proposition then follows since $J(1) = (d\exp_p)_v(w)$.

Prop 21.2: If $\gamma : [0,1] \longrightarrow M$ is a geodesic from p to q, w/ some $t_0 \in (0,1)$ s.t. $\gamma(t_0) \in \mathrm{Conj}_\gamma p$.

Then $\exists$ $X \in T_\gamma \Omega(p,q)$ s.t. $I_\gamma(X, X) < 0$.

In particular, γ is not minimizing.

Pf: $\exists$ $J \in \mathcal{J}(\gamma|_{[0,t_0]})$ s.t. $J(0) = 0$, $J(t_0) = 0$.

Let

$$V(t) := \begin{cases} J(t), & 0 \leq t \leq t_0 \\ 0, & t \in [t_0, 1] \end{cases}$$

Let W be a proper normal v.f along γ s.t.

$$W(t_0) = \Delta_{t_0} D_t V.$$

Note $\Delta_{t_0} D_t V = - D_t J(t_0) \neq 0$ since otherwise J would be trivial.

For $\varepsilon \ll 1$, consider $X_\varepsilon = V + \varepsilon W$. Then

$$I(X_\varepsilon, X_\varepsilon) = I(V, V) + 2\varepsilon I(V, W) + \varepsilon^2 I(W, W)$$

V satisfies Jacobi-eqn on $[0, t_0]$, $[t_0, 1]$. So

$$I(V, V) = - \langle \Delta_{t_0} D_t V, V(t_0) \rangle = 0$$

$$I(V, W) = - |W(t_0)|^2$$

So,

$$I(X_\varepsilon, X_\varepsilon) = -2\varepsilon |W(t_0)|^2 + \varepsilon^2 I(W,W)$$

$$= -\varepsilon\left(2|W(t_0)|^2 - \varepsilon I(W,W)\right)$$

Since $|W(t_0)|^2 > 0$, $\exists\ \varepsilon << 1$ s.t $\varepsilon I(W,W) < 2|W(t_0)|^2$. Done!

There is a far reaching generalization.

Def^n: 1). The multiplicity of a conjugate pt $q \in \mathrm{Conj}_\gamma p$ is the dimension of the sp. of Jacobi fields vanishing at p & q. In other words, if $q = \exp_p v$, then multiplicity

$$\nu_\gamma(q) = \dim \ker (d\exp_p)_v$$

2). The index of a geodesic γ is

$$\lambda(\gamma) := \{X \in T_\gamma \Omega \mid I_\gamma(X,X) < 0\}.$$

Th^m 21.3 (Morse) The index $\lambda(\gamma)$ equals the number of points $\gamma(t)$, $0 < t < 1$ s.t. $\gamma(t) \in \mathrm{Conj}_\gamma p$. & each $\gamma(t)$ is counted w/ multiplicity. The index $\lambda(\gamma)$ is always finite.

• Jacobi Comparison Theorem

Lemma 21.4 Sps. $u, v \in C^2(0,T)$, $u \in C^0[0,T]$, & $u > 0$. Sps $u > ($

&

$$\begin{cases} \ddot{u}(t) + a(t)u(t) = 0 \\ \ddot{v}(t) + a(t)v(t) \geq 0 \\ u(0) = v(0) = 0, \ \dot{u}(0) = \dot{v}(0) > 0 \end{cases}$$

for some $a \in C^0[0,T]$. Then $v(t) \geq u(t)$ on $[0,T]$.

Pf: Consider $f(t) = v(t)/u(t)$

L'Hospital $\Rightarrow \lim_{t \to 0^+} f(t) = 1$.

Claim $f' \geq 0$.

Pf: $f'(t) = \dfrac{\dot{v}u - \dot{u}v}{u^2}$.

Now $\dot{v}u - \dot{u}v\,(0) = 0$.

Also $(\dot{v}u - \dot{u}v)' = \ddot{v}u + avu \geq 0$.

So $\dot{v}u - \dot{u}v \geq 0$ on $[0,T)$.

Hence $f' \geq 0$

Def^n: We say that a Riemannian mfld (M,g) has sectional curvature bounded below (resp. above) by $K \in \mathbb{R}$, and write this as $\sec_g \geq K$ (resp. $\sec_g \leq K$) if $\forall p \in M$, $\forall$

2-dim planes $\Pi \subset T_pM$ we have

$$\sec(\Pi) \geq K \quad (\text{resp. } \sec(\Pi) \leq K).$$

<u>Th</u>^{m} 21.5 (Jacobi field comparison) Sps (M, g) s.t $\sec_g \leq K$. If γ is a unit speed geodesic & $J \in \mathcal{J}^\perp(\gamma)$ s.t $J(0) = 0$, then

$$|J(t)| \geq \begin{cases} t\,|D_t J(0)|, & 0 \leq t \quad \text{if } K = 0 \\ R \sin\frac{t}{R} \cdot |D_t J(0)|, & 0 \leq t \leq \pi R, \\ & K = 1/R^2 \\ R \sinh\frac{t}{R}\, |D_t J(0)|, & t \geq 0, \\ & K = -1/R^2. \end{cases}$$

<u>Pf</u>: Note $|J(t)|$ is smooth whenever $J(t) \neq 0$.

$$\frac{d}{dt}|J(t)| = \frac{\langle D_t J, J\rangle}{\langle J, J\rangle^{1/2}}.$$

$$\frac{d^2}{dt^2}|J(t)|^2 = \frac{\langle D_t^2 J, J\rangle}{|J|} + \frac{|D_t J|^2}{|J|} - \frac{\langle D_t J, J\rangle^2}{|J|^3}.$$

Cauchy Schwartz $\Longrightarrow$ $|D_t J|^2 |J|^2 \geq \langle D_t J, J\rangle^2$.

Jacobi eq $\Longrightarrow$ $\langle D_t^2 J, J\rangle = -Rm(J, \dot{\gamma}, \dot{\gamma}, J)$.

So
$$\frac{d^2}{dt^2}|J| \geq -\frac{Rm(J,\dot{r},\dot{r},J)}{|J|}$$

Now, $\langle J, \dot{r}\rangle = 0$. Also $|\dot{r}|^2 = 1$. So

$$Rm(J,\dot{r},\dot{r},J) = \sec(J\wedge\dot{r})\,|J|^2$$
$$\leq K|J|^2.$$

So
$$\boxed{\frac{d^2}{dt^2}|J| \geq -K|J|}$$

Recall that if $u(t) = \begin{cases} R\sin\frac{t}{R}, & K = 1/R^2 \\ t, & K = 0 \\ R\sinh t/R & K = -1/R^2 \end{cases}$

Then $\ddot{u} + Ku = 0$. To apply Sturm comparison, we need. $|J|'(0) = 1$ since $\dot{u}(0) = 1$. W.l.o.g, assume $|D_t J(0)| = 1$.

If $D_t J(0) = W$, then near $t=0$ in normal coordinate $J(t) = tW^i\partial_i$. So

$$\left.\frac{d}{dt}\right|_{t=0} |J(t)| = \lim_{t\to 0}\frac{|J(t)| - |J(0)|}{t}$$
$$= \lim_{t\to 0}|W|_{g(t)} = |W|_{g(0)} = |D_t J(0)|$$
$$= 1.$$

Then Sturm comparison $\Rightarrow |J|(t) > u(t)$ as long as $u(t) > 0$. This proves the theorem.

Cor 21·6 (Conjugate pt Comparison). Sps. $\sec_g \leq K$. Then

1) $K \leq 0 \Rightarrow$ there is no conjugate pt along any geodesic. i.e $\exp_p : T_pM \to M$ is a local diffeo.

2) $K = 1/R^2$, then the first conjugate pt. occurs at a dist $\geq \pi R$.

Pf: 1) $K \leq 0 \Rightarrow |J(t)| > 0 \quad \forall\, t>0$ by Th^m 21·6.

2) $K = 1/R^2 > 0$, then $|J(t)| \geq R \sin\frac{t}{R} \cdot |D_t J(0)|$

The first zero of $\sin t/R$ is at $t = \pi R$.

So $|J(t)| > 0 \quad \forall \quad t < \pi R$.

Rk: This result is sharp. $M = S^n_R$, then the first conjugate pt occurs exactly at a distance of πR.

Cor 21·7 (Metric Comparison). Again if $\sec_g \leq K$. Then in any normal neighbourhood

$$g(V, V) \geq g_K(V, V),$$

where g_K is the metric $dr^2 + A_K(r)\, g_{S^{n-1}}$, where

A_K is as in the previous lecture

Pf: Decompose $V = V^T + V^\perp$, where V^T is tangent to a geodesic sphere, $V^\perp$ tangent to radial geodesics. Then

$$g(V, V) = g(V^T, V^T) + g(V^\perp, V^\perp).$$

As before $g(V^\perp, V^\perp) = g_K(V^\perp, V^\perp)$.

Also $V^T = J(1)$ for some J s.t $J = 0$ at $t = 0$. So Th^m $\Longrightarrow$ $g(V^\perp, V^\perp) \geq g_K(V^\perp, V^\perp)$.

$D_t J(0) = V^T$.