LECTURE - 22

• COVERING MAPS AND LOCAL ISOMETRIES

Def^n: Let M, N be Hausdorff, p. connected, locally p. conn. topological spaces. A cont. surjective map $f : N \longrightarrow M$ is said to be a <u>covering map</u> if $\forall\ p \in M$, $\exists$ a path connected open nbd V of p s.t. $f^{-1}(V) = \bigsqcup_{\alpha \in I} U_\alpha$.

w/ (1) $U_\alpha \cap U_\beta = \phi$ if $\alpha \neq \beta$

(2) $f|_{U_\alpha} : U_\alpha \longrightarrow V$ is a homeomorphism.

We then say that (N, f) is a covering space of M.

Def^n We say that (N, f) is a <u>universal cover</u> if for any other cover (X, π), $\exists$ a covering map $h : N \longrightarrow X$ s.t $f = \pi \circ h$.

$$\begin{tikzcd}
 & N \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\
X \arrow[r, "\pi"] & M
\end{tikzcd}$$

<u>Example</u>: Define $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ by $f(z) = z^n$. Then f is a covering map, but $\mathbb{C}^*$ is <u>not</u> its own universal cover. In fact the universal cover is $(\mathbb{C}, \exp)$.

Similarly $(\mathbb{R}, \exp)$ is the universal cover of S^1.

<u>Prop 22.1</u> (N, f) is a universal cover of M iff it is a cover & N is <u>simply connected</u>. Moreover if (N', f') is another universal cover, then $\exists$ a homeo $\phi : N \to N'$

<u>Recall</u> $\pi_1(M, p) = \{$homotopy classes of loops at $p \in M\}$

can be given the structure of a group. M path conn. $\Longrightarrow \pi_1(M, p) \cong \pi_1(M, q)$. So we write this common group as $\pi_1(M)$ & call it the <u>fundamental grp</u>. We say M is simply connected if $\pi_1(M) = \{id\}$

<u>Rk</u>: If (N, f) is a covering sp. of M, then f is a local homeo. Converse is not true. Consider $f : (0, \infty) \to S^1$, $f(x) = e^{2\pi i x}$. Then f is a local homeo but not a covering map.

<u>Def^n</u>: Let M, N be manifolds & $f : N \to M$ a covering map. We say, f is a <u>smooth covering map</u> if f is smooth & $\forall\, p \in M$, $\exists$ nbd V s.t. $f^{-1}(V) = \bigsqcup_\alpha U_\alpha$ where $f : U_\alpha \to V$ is a diffeo

Prop 22.2 Let (N, f) be a cover of M. If M is a manifold, $\exists !$ smooth structure on N s.t (N, f) is a smooth covering sp of M.

Th^m 22.3 Let M be any (conn.) manifold. Then $\exists !$ (upto diffeos) smooth universal cover $(\tilde{M}, f)$ s.t. $f: \tilde{M} \longrightarrow M$ is a smooth covering map.

In Riemannian geometry the foll. theorem helps in constructing covers.

Th^m 22.4 (Ambrose). Let (M, g) & (N, h) be Riemannian mfds & $f: N \to M$ be a local isometry (i.e $f^*g = h$). If (N, h) is complete, then

(1) (M, g) is complete

(2) f is a smooth covering map.

Pf: **Claim 1** (Geodesic lifting). Let $\gamma: [a,b] \longrightarrow M$ be a geodesic. For any $p \in f^{-1}(\gamma(a))$, $\exists !$ "lift" to a geodesic $\tilde{\gamma}: [a,b] \longrightarrow M$ s.t $\gamma(t) = f(\tilde{\gamma}(t))$ and $\tilde{\gamma}(a) = p$.

Pf: Note $df_p: T_pN \longrightarrow T_{f(p)}M$ is a linear isometry. Let $\tilde{V} := df_p^{-1}(\dot{\gamma}(a))$, and

$$\tilde{\gamma}(t) = \exp_p (t-a)\tilde{V}$$

(N, h) complete $\Longrightarrow$ $\tilde{\gamma}$ exists on $[a, b]$.
f local isom $\Longrightarrow$ $\rho(t) := f \circ \tilde{\gamma}$ is a geodesic.
starting at $f(p) = \gamma(a)$ w/ $\rho'(a) = df_p(\tilde{v}) = \gamma'(a)$
Uniqueness $\Rightarrow$ $\rho = \gamma$ i.e $\gamma = f \circ \tilde{\gamma}$

Claim 2: f is surjective

Pf: f local isom $\Longrightarrow$ f is local diffeo (by inverse function theorem) $\Longrightarrow$ $f(N)$ is open. Sps $f(N)$ is not closed. Let $q \in \overline{f(N)} \setminus f(N)$. Let $B_\delta(x)$ be a normal nbd. Then $\exists$ $x \in B_\delta(x) \cap f(N)$. Let $\gamma : [0, \varepsilon]$ be the ! unit sp. geodesic from x to q, and let $\tilde{\gamma}$ be its unique lift w/ $\tilde{\gamma}(0) = \tilde{x}$ s.t $f(\tilde{x}) = x$. But then $\gamma(\varepsilon) = f(\tilde{\gamma}(\varepsilon))$. So $q = f(\tilde{\gamma}(\varepsilon)) \in f(N)$. Contradiction! So $f(N)$ is closed. M conn. $\Longrightarrow$ $f(N) = M$.

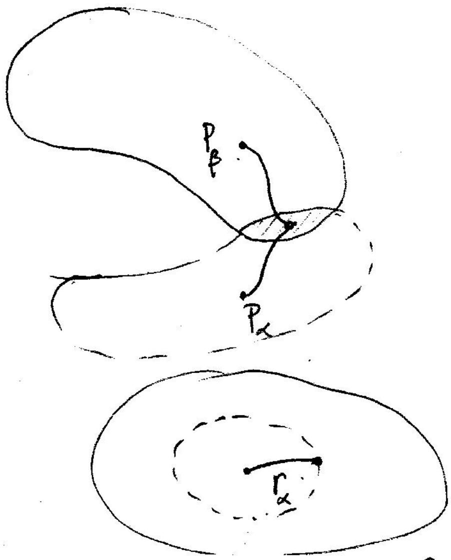
Claim 3: (M, g) is complete.

Pf: Let $\gamma : I \longrightarrow \mathbb{R}$ be a geodesic in M. Sps $0 \in I$ & $\gamma(0) = p$. f surj $\Longrightarrow$ $\exists$ a $\tilde{p}$ s.t $f(\tilde{p}) = p$. Let $\tilde{\gamma}$ be the unique lift s.t $\tilde{\gamma}(0) = \tilde{p}$. N complete $\Longrightarrow$ $\tilde{\gamma} : \mathbb{R} \longrightarrow N$. Then $\gamma = f \circ \tilde{\gamma}$ is also defined on all of $\mathbb{R}$.

All we need to show is that f is a ${}^{\text{smt}}$ covering map. Let $p \in M$ & $V := B_\delta(p)$ be a normal nbd. Let $f^{-1}(p) = \{p_\alpha \mid \alpha \in I\}$, and set $U_\alpha = B_\delta(p_\alpha)$.

Claim 4: $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$.

Pf:



Sps $\tilde{q} \in U_\alpha \cap U_\beta$ & set $q = f(\tilde{q})$. Let $\tilde{r}_\alpha$ & $\tilde{r}_\beta$ be ! min. geodesic's from $\tilde{q}$ to p_α & p_β resp.

Then $r_\alpha := f \circ \tilde{r}_\alpha$ & $r_\beta := f \circ \tilde{r}_\beta$ are 2 minimal geodesics from q to p. Contradiction!

Claim 5: $f^{-1}(V) = \bigsqcup_{\alpha \in I} U_\alpha$

Pf: Let $\tilde{x} \in f^{-1}(V)$. Then $x := f(\tilde{x}) \in V$ so $\exists !$ min unit speed geodesic $r : [0, \varepsilon] \longrightarrow M$ s.t. $r(0) = x$, $r(\varepsilon) = p$. Here $\varepsilon = d(x, p) < \delta$. Let $\tilde{r}$ be the unique lift s.t. $\tilde{r}(0) = \tilde{x}$. Then $f(\tilde{r}(\varepsilon)) = r(\varepsilon) = p$, so $\tilde{r}(\varepsilon) = p_\alpha$ for some $\alpha \in I$.

Also $f^* g = h \implies \varepsilon = L_g(r) = L_h(\tilde{r}) < \delta$. So $\tilde{x} \in U_\alpha$.

Claim 6: $f : U_\alpha \longrightarrow V$ is a diffeo.

Pf: f local isom $\Longrightarrow$ $\forall\ X_\alpha \in T_{p_\alpha}N$,

$$f(\exp_{p_\alpha} X_\alpha) = \exp_p df_{p_\alpha}(X_\alpha).$$

Note that p_α has no conjugate pts in U_α. Also for every pt in U_α, $\exists!$ geod. from p_α to that pt, since the same is true for p & $V = B_\delta(p)$. Hence $\exp_{p_\alpha}$ & $\exp_p$ are diffeos on $\tilde{B}_\delta(0) \subset T_{p_\alpha}N$ & $B_\delta(0) \subset T_pM$ resp. Since df is a linear isom which is a diffeo on all of $T_{p_\alpha}N$,

$$f = \exp_p \circ df_{p_\alpha} \circ \exp_{p_\alpha}^{-1}.$$

is a diffeo from U_α onto V.

Cor 22.5 (Cartan–Hadamard). Let (M, g) be a complete Riemannian mfld w/ $\sec_g \leq 0$. Then for any $p \in M$, $\exp_p : T_pM \longrightarrow M$ is a smooth covering map. In particular if M is simply connected, then $\exp_p$ is a diffeomorphism.

Pf: By conjugate pt. comparison, there are no conjugate pts. So, $\exp_p$ is a local diffeo. Let $h := \exp_p^* g$. Then $\exp_p : (T_pM, h) \longrightarrow (M, g)$ is a local isom (by def^n).

Claim: (T_pM, h) is complete

Pf: h-Geodesics on T_pM passing through 0 are st-line. So, (T_pM, h) is complete by Hopf-Rinow.

So Th^m 2.4 $\Longrightarrow$ $\exp_p$ is a smt. covering.

If M is s-c, then $id: M \to M$ is a universal covering. So $\exists$ covering map $f: M \to T_pM$ s.t. $\exp_p \circ f = id$. But then f is injective & hence f is a diffeo. So $\exp_p$ is also a diffeo.

• Characterization Of Space Forms

Th^m 22.6: Let (M, g) complete be s.t $\sec_g \equiv K$. If M is simply connected then $(M,g) \cong (\tilde{M}_K, \tilde{g}_K)$ where

$$(\tilde{M}_K, \tilde{g}_K) \cong \begin{cases} (S^n_R, g_{S^n_R}), & K = 1/R^2 > 0 \\ (\mathbb{R}^n, g_{\mathbb{R}^n}), & K \equiv 0 \\ (\mathbb{H}^n_R, g_{\mathbb{H}^n_R}), & K = -1/R^2 < 0 \end{cases}$$

model sp.

Cor 22.7: Sps (M, g) is a complete (conn.) Riem. mfld s.t $\sec_g \equiv K$. Then $(M, g) \cong (\tilde{M}_K/\Gamma, \tilde{g}_K)$ where $(\tilde{M}_K, \tilde{g}_K)$ are the model spaces above & Γ is a discrete sub-group of $\mathrm{Isom}(\tilde{M}_K, \tilde{g}_K)$ s.t $\Gamma \cong \pi_1(M)$.

Pf: If $f: \tilde{M} \longrightarrow M$ is the (smt) universal cover, then f is a local diffeo, so induces a metric $\tilde{g} = f^* g$ on $\tilde{M}$ w/ $\sec_{\tilde{g}} \equiv K$ Thm $\Longrightarrow$

$(\tilde{M}, \tilde{g}) \cong (\tilde{M}_K, \tilde{g}_K)$. Let Γ be the Deck trans. grp

$$\Gamma = \{\varphi : \tilde{M} \longrightarrow \tilde{M} \text{ homeo s.t. } f \circ \varphi = f\}.$$

General theory $\Longrightarrow$ $\Gamma \cong \pi_1(M)$ & $M \cong_{diff.} \tilde{M}/\Gamma$

__Claim 1__: $\Gamma \subset \mathrm{Isom}(\tilde{M}, \tilde{g})$.

__Pf__: If $\varphi \in \Gamma$. Then since $f \circ \varphi = f$.

$$\varphi^* \tilde{g} = \varphi^* f^* g = f^* g = \tilde{g}.$$

So $(M, g) \cong (\tilde{M}/\Gamma, \tilde{g}) \cong (\tilde{M}_K/\Gamma, \tilde{g}_K)$.

__Claim 2__: Γ is discrete.

__Pf__: Sps $\varphi_i \in \Gamma$ s.t. φ_i has a l.p in $\mathrm{Isom}(\tilde{M}, \tilde{g})$

Since $\Gamma \curvearrowright \tilde{M}$ is fixed pt free, $\forall \tilde{p} \in \tilde{M}$, $\varphi_i(\tilde{p})$ is an infinite set & has a l.p in $\tilde{M}$. But this is impossible, since $f \circ \varphi_i(\tilde{p}) = p \ \forall \ i$.

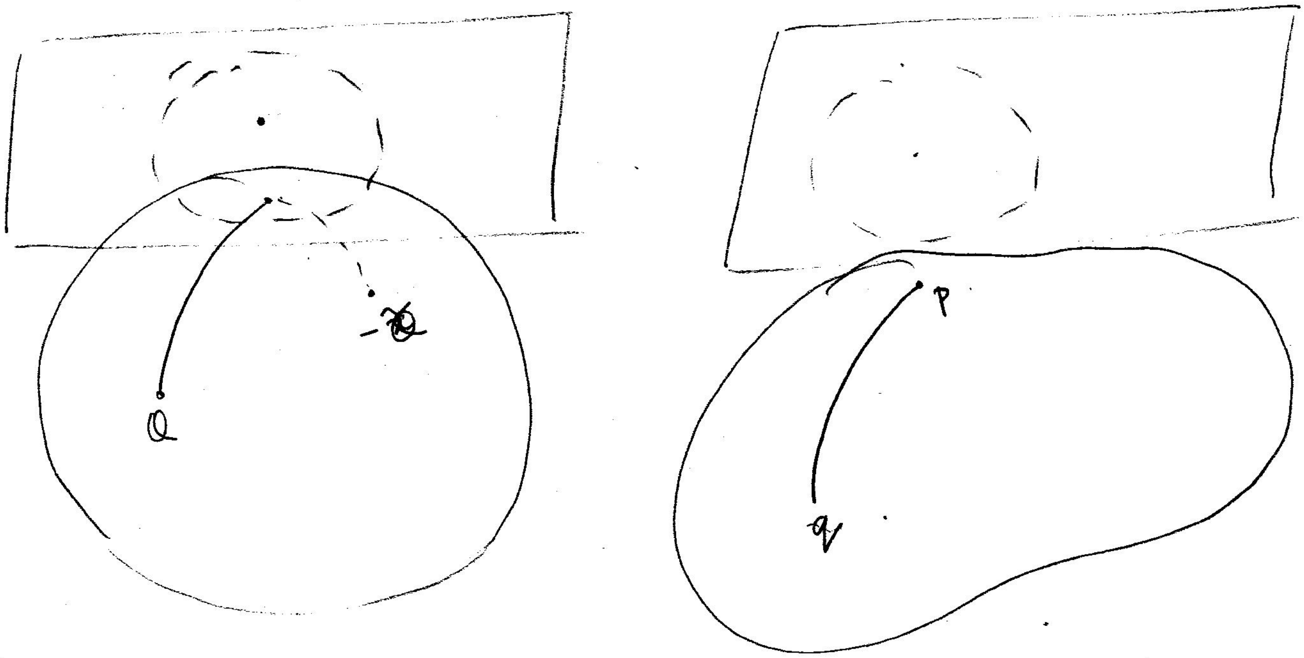
__Pf of Thm__: __CASE I__ $K \leq 0$. Then Cartan-Hadamard $\Longrightarrow$ $\exp_p : T_pM \longrightarrow M$ is a diffeo. So $\tilde{g} := \exp_p^* g$ is a const. sectional curv. metric.

Also, Euclidean coordinates on T_pM are normal coordinates for $\tilde{g}$ since geodesic through 0 are st lines. So $(T_pM, \tilde{g}) \cong (\tilde{M}_K, \tilde{g}_K)$.

CASE 2: $K = 1/R^2 > 0$. Note $\exp_N : B_{\pi R}(0) \longrightarrow S^n_R \setminus \{-N\}$ is a diffeo. If $p \in M$, conj. pt comp $\Longrightarrow$

$\exp_p : B^g_{\pi R}(0) \subset T_pM \longrightarrow M$ is also a local diffeo.

Let $L : T_N S^n_R \longrightarrow T_pM$ be <u>any</u> linear isom.



Consider $\exp_p \circ L : B_{\pi R}(0) \longrightarrow \exp_p(B^g_{\pi R}(0))$.

Then $(\exp_p \circ L)^* g$ & $(\exp_N^* g_{S^n_R})$ are both sec $\equiv 1/R^2$ metrics on $B_R(0) \subset T_N S^n_R$ & Euclidean coordinates are normal coordinate. So $\exists$ a map

$$\phi : S^n_R \setminus \{-N\} \longrightarrow M$$

$$\phi = \exp_p \circ Q \circ \exp_N^{-1}$$

is a local isom.

Now let $Q \in S^n_R \setminus \{-N, N\}$ & $q = \phi(Q)$,

$$L_1 := \phi_* : T_q S^n_R \longrightarrow T_q M.$$

Then $|||^{ly}$ $\exists$ local isom map $\phi_1 : S^n \setminus \{-Q\} \longrightarrow M$ s.t.

$\phi_1 = \exp_q \circ L_1 \circ \exp_Q$, $\phi_1(Q) = q$. $(\phi_r)_*|_q = \phi_*|_q$.

Since ϕ, ϕ_1 are local isom & $S^n \setminus \{N \cup -Q\}$ is connected, $\phi \equiv \phi_1$. So ϕ extends to a local isom $\phi : S^n \longrightarrow M$. M comp $\Longrightarrow$ ϕ is a covering. M s.c $\Longrightarrow$ ϕ is a diffeo, & hence isom.