LECTURE - 23

• RAUCH COMPARISON THEOREM.

<u>Th^m</u> 23.1 (Rauch). Let (M^n, g) & $(\tilde{M}^n, \tilde{g})$ be Riemannian mflds, & let $\gamma, \tilde{\gamma} : [0, l] \longrightarrow M, \tilde{M}$ be unit speed geodesics. Set $\gamma'(t) = T$ & $\tilde{\gamma}'(t) = \tilde{T}$. Let J & $\tilde{J}$ be Jacobi fields along γ & $\tilde{\gamma}$ resp. Sps further

(1) $\tilde{\gamma}$ has no conjugate points on $[0, l]$.

(2) For any $X_t, \tilde{X}_t \in T_{\gamma(t)}M, T_{\tilde{\gamma}(t)}M$,

$$\sec_{\tilde{g}}(\tilde{X}_t \wedge \tilde{T}) \geq \sec_g(X_t \wedge T).$$

(3) (a) $J(0) = 0$, $\tilde{J}(0) = 0$ (b) $|J'(0)| = |\tilde{J}'(0)|$

(c) $\langle T, J'(0) \rangle = \langle \tilde{T}, \tilde{J}'(0) \rangle$.

Then $|J(t)| \geq |\tilde{J}(t)|$ $\forall$ $t \in [0, l]$. In particular γ has no conj. points on $[0, l]$.

<u>Rk</u>: 1) If $(\tilde{M}, \tilde{g}) = (\tilde{M}_K, \tilde{g}_K)$ is the sp form w/ $\sec_{\tilde{g}} \equiv K$, and $\sec_g \leq K$. If γ is a normal geodesic & $J \in \mathcal{J}^{\perp}(\gamma)$ w/ $J(0) = 0$, $|J'(0)| = 1$ & if J_K is a corresponding Jacobi field in $(\tilde{M}_K, \tilde{g}_K)$, then

$$|J(t)| \geq |\tilde{J}(t)| = \begin{cases} R \sin \frac{t}{R}, & 0 \leq t \leq \pi R \quad K = 1/R^2 \\ t, & t \geq 0, \quad K = 0 \\ R \sinh \frac{t}{R}, & t \geq 0 \quad K = -\frac{1}{R^2} \end{cases}$$

So we recover Jacobi comparison.

2) If inequality in (2) is strict, then

$$|J(t)| > |\tilde{J}(t)|.$$

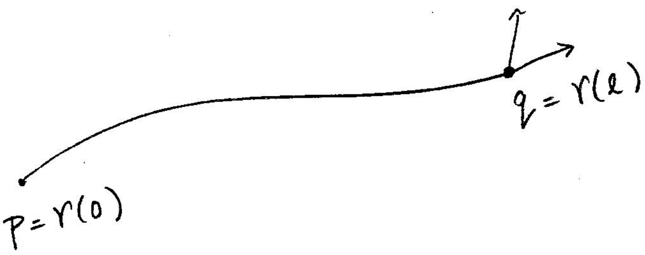
For the proof we need the following

<u>Lemma</u> 23.2 Sps $\gamma : [0, \ell] \longrightarrow M$ is a unit speed geodesic s.t $p = \gamma(0)$ has no conjugate pt along γ. If J is a Jacobi field along γ & $X \in \mathcal{J}(\gamma)$ s.t $X(0) = J(0) = 0$ and $X(\ell) = J(\ell)$. Then

$$I_\gamma(J, J) \leq I_\gamma(X, X)$$

w/ equality $\Longleftrightarrow$ $X = J$.

<u>Pf</u>:



Let $\{E_i\}$ be o.n.b for $T_q M$, & extend each E_i to be Jacobi fields along γ w/ $E_i(0) = 0$. This is possible since $q \notin \mathrm{Conj}_\gamma(p)$. Each $\{E_i(t)\}$ is a basis for $T_{\gamma(t)}M$ except at p. Since $E_i(0) = 0$, we can write $E_i(t) = t V_i(t)$. Then $E_i'(0) = V_i(0)$, so V_i l.i on all of $[0, \ell]$. So $\exists$ functions $a_i : [0, \ell] \to \mathbb{R}$ s.t

$$X = \sum a_i(t) V_i(t).$$

Since $X(0) = 0$, $f_i(t) = a_i(t)/t$ is a piecewise smooth function & we can write

$$X = \sum f_i E_i$$

Note that $J = \sum f_i(\ell) E_i$ since $\sum f_i(\ell) E_i(\ell) = X(\ell)$
$= J(\ell)$

and $\sum f_i(\ell) E_i(0) = 0$ & $\sum f_i(\ell) E_i(\cdot)$ is a Jacobi field.

Claim 1: $I_\gamma(J, J) = \langle J'(\ell), J(\ell) \rangle = \sum_{i,j} f_i(\ell) f_j(\ell) \langle E_i'(\ell), E_j(\ell) \rangle$

Pf: Since $J'' = -R(J, \dot{\gamma})\dot{\gamma}$,

$$I_\gamma(J, J) = \int_0^\ell \langle J', J' \rangle - Rm(J, \dot{\gamma}, \dot{\gamma}, J)$$

$$= \int_0^\ell \frac{d}{dt} \langle J', J \rangle$$

$$= \langle J'(\ell), J(\ell) \rangle \quad \text{since } J(0) = 0$$

Claim 2: $\langle E_i', E_j \rangle = \langle E_i, E_j' \rangle$

Pf: Since each E_i is a Jacobi field

$$\frac{d}{dt}\left(\langle E_i', E_j \rangle - \langle E_i, E_j' \rangle \right) = \langle E_i'', E_j \rangle - \langle E_i, E_j'' \rangle$$

$$= -Rm(E_i, \dot{\gamma}, \dot{\gamma}, E_j) + Rm(E_j, \dot{\gamma}, \dot{\gamma}, E_i)$$

$$= 0$$

So $\langle E_i', E_j \rangle - \langle E_i, E_j' \rangle = C_{ij}$ ← const. in 't'.

Plugging in $t = 0$, $C_{ij} = 0 \ \forall i, j$.

We now compute

$$D_t X = \sum f_i' E_i + \sum f_i E_i' = A + B.$$

$$I_\gamma(X, X) = \int \langle A, A \rangle + \langle B, B \rangle + \langle A, B \rangle + \langle B, A \rangle - Rm(X, \dot{\gamma}, \dot{\gamma}, X)$$

Now,

$$\int \langle B, B\rangle = \sum_{i,j} \int_0^l f_i f_j \langle E_i', E_j'\rangle =$$

$$= \sum_{i,j} \int_0^l f_i f_j \frac{d}{dt}\langle E_i', E_j\rangle - \sum_{i,j}\int_0^l f_i f_j \langle E_i'', E_j\rangle$$

$$= \sum_{i,j} f_i(l) f_j(l)\langle E_i'(l), E_j(l)\rangle - \sum_{i,j}\int_0^l f_i' f_j \langle E_i', E_j\rangle$$

$$- \sum_{i,j}\int_0^l f_i f_j' \langle E_i', E_j\rangle + \sum_{i,j}\int_0^l f_i f_j Rm(E_i, \dot{r}, \dot{r}, E_j)$$

$$= I_r(J, J) - \int\langle A, B\rangle - \int\langle B, A\rangle$$

$$+ \int_0^l Rm(X, \dot{r}, \dot{r}, X)$$

So $I_r(X, X) = I_r(J, J) + \int\langle A, A\rangle \geq I_r(J, J)$.

Equality $\Leftrightarrow$ $A = 0 \Leftrightarrow f_i'(t) = 0 \ \forall\ t$ i.e $f_i(t) = f_i(l)\ \forall\ t \Leftrightarrow X = J$.

Pf of Rauch's Th^m First, sps J & $\tilde{J}$ are normal fields. Define $u(t) = |J(t)|^2$, $\tilde{u}(t) = |\tilde{J}(t)|^2$. Note

$$u'(t) = 2\langle J'(t), J(t)\rangle, \quad u''(t) = 2|J'(t)|^2 + 2\langle J''(t), J(t)\rangle$$

And so by L'Hospital,

$$\lim_{t\to 0^+}\frac{u(t)}{\tilde{u}(t)} = \lim_{t\to 0^+}\frac{u''(t)}{\tilde{u}''(t)} = \lim_{t\to 0^+}\frac{|J'(0)|^2}{|\tilde{J}'(0)|^2} = 1.$$

Goal: $u(t) \geq \tilde{u}(t)\ \forall\ t$.

Enough to show that $u/\tilde{u}$ is $\uparrow$ or $\iff$
$(u/\tilde{u})' \geq 0 \iff u'/u \geq \tilde{u}'/\tilde{u}$.

Since $\tilde{\gamma}$ has no conj. point, $\tilde{u} > 0 \ \forall t$.
Let $b = \sup\{t \mid u(t) > 0\}$. For any $a < b$, define

$$X_a(t) = \frac{J(t)}{|J(a)|} \ ; \quad \tilde{X}_a(t) = \frac{\tilde{J}(t)}{|\tilde{J}(a)|}$$

<u>Claim</u> $I_{\tilde{\gamma}|_{[0,a]}}(\tilde{X}_a, \tilde{X}_a) \leq I_{\gamma|_{[0,a]}}(X_a, X_a)$.

Assuming this, we note

$$I_\gamma(X_a, X_a) = \int_0^a \langle X_a', X_a' \rangle - Rm(X_a, \dot{\gamma}, \dot{\gamma}, X_a)$$

$$= \int_0^a \frac{d}{dt}\langle X_a', X_a \rangle = \langle X_a'(a), X_a(a) \rangle.$$

and so Claim $\Rightarrow \langle \tilde{X}_a'(a), \tilde{X}_a(a) \rangle \leq \langle X_a'(a), X_a(a) \rangle$

So,

$$\frac{\tilde{u}'(a)}{\tilde{u}(a)} = \frac{2\langle \tilde{J}'(a), \tilde{J}(a) \rangle}{|\tilde{J}(a)|^2} = 2\langle \tilde{X}_a'(a), \tilde{X}_a(a) \rangle$$

$$\leq 2\langle X_a'(a), X_a(a) \rangle = \frac{u'(a)}{u(a)}.$$

So $\forall t \in [0,b)$, $u'(t)\tilde{u}(t) - \tilde{u}'(t)u(t) \geq 0$ & so
$\forall t \in [0,b)$, $|\tilde{J}(t)| \leq |J(t)|$. If $b < \ell$, then
$u(b) = 0$. By continuity, $\tilde{J}(b) = 0$ contradicting
the fact that $\tilde{\gamma}$ has no conj. point. So $b = \ell$.
Moreover $J(\ell) \neq 0$ by the same reason.

More generally, we decompose $J(t) = J^{\perp}(t) + \langle T, J\rangle T$, $\tilde{J} = \tilde{J}^{\perp} + \langle \tilde{J}, \tilde{T}\rangle \tilde{T}$. Recall that $\langle J, T\rangle$ is a linear function of t, & since $\frac{d}{dt}\langle J, T\rangle = \langle J', T\rangle$, we have

$$\langle T, J\rangle = \langle T(0), J(0)\rangle + t\langle T(0), J'(0)\rangle$$
$$= \langle \tilde{T}, \tilde{J}\rangle.$$

by (a) & (c). So $|J|^2 - |J^{\perp}|^2 = |\tilde{J}|^2 - |\tilde{J}^{\perp}|^2$.
One can check that $J^{\perp}$ & $\tilde{J}^{\perp}$ also satisfy (a)-(c).
By above discussion $|J^{\perp}|^2 \geq |\tilde{J}^{\perp}|^2$ & so $|J|^2 \geq |\tilde{J}|$

<u>Pf of Claim</u>: Let $P_{-\gamma}$ be $\|$-transport along $-\gamma$ & $P_{\tilde{\gamma}}$ be $\|$-transp along $\tilde{\gamma}$. Let $\mathcal{I} : T_{\gamma(0)}M \longrightarrow T_{\tilde{\gamma}(0)}\tilde{M}$ be an isometry, and define $\mathcal{I}_t : T_{\gamma(t)}M \longrightarrow T_{\tilde{\gamma}(t)}M$ by $I_t = P_{\tilde{\gamma}} \circ \mathcal{I} \circ P_{-\gamma}$. Let

$$\tilde{Y}_a := I_t(X_a)$$

More explicitly, let $\{E_i^{(0)}\}$ be an o.n.b for $T_{\gamma(0)}M$ s.t $E_1^{(0)} = T(0)$, $E_2^{(0)} = X_a(0)$ & let $\{E_i(t)\}$ be $\|$-transp. along γ. $\|^{ly}$ we let $\{\tilde{E}_i(t)\}$ o.n.b for $T_{\tilde{\gamma}(t)}\tilde{M}$ s.t $\tilde{E}_1(0) = \tilde{T}(0)$ & $\tilde{E}_2(0) = \tilde{X}_a(0)$. If $X_a(t) = X_a^i(t) E_i(t)$, then we can take

$$\tilde{Y}_a = X_a^i(t) \cdot \tilde{E}_i(t).$$

Note $E_1(t) = T$, $\tilde{E}_1 = \tilde{T}$.

We can easily check that

$$\left.\begin{aligned} |\tilde{Y}_a|^2 &= \sum_i X_a^i(t)^2 = |X_a|^2 = 1. \\ \langle \tilde{Y}_a'(t), \tilde{Y}_a'(t) \rangle &= \langle X_a'(t), X_a'(t) \rangle. \end{aligned}\right\} (*).$$

Also, by hypothesis on sectional curv, since T & X_a are o.n,

$$Rm(X_a, T, T, X_a) = \sec_g(X_a \wedge T) \leq \sec_{\tilde{g}}(\tilde{Y}_a \wedge \tilde{T}).$$
$$= \widetilde{Rm}(\tilde{Y}_a, \tilde{T}, \tilde{T}, \tilde{Y}_a).$$

By Lemma 23.2,

$$I_{\tilde{\gamma}}(\tilde{X}_a, \tilde{X}_a) \leq I_{\tilde{\gamma}}(\tilde{Y}_a, \tilde{Y}_a) = \int_0^a \langle \tilde{Y}_a', \tilde{Y}_a' \rangle - \widetilde{Rm}(\tilde{Y}_a, \tilde{T}, \tilde{T}, \tilde{Y}_a)$$

$$\leq \int_0^a \langle X_a'(t), X_a'(t) \rangle - Rm(X_a, T, T, X_a).$$

$$= I_\gamma(X_a, X_a).$$

· Applications of Rauch's Theorem.

Cor 23.3 Let (M, g) be a complete Riem. mfld s.t

$$0 < C_1 \leq \sec_g \leq C_2.$$

Let $\gamma: [0, \ell] \to M$ be a unit-speed geodesic s.t. $q = \gamma(\ell)$ is the first point along γ conj. to $p = \gamma(0)$. Then

$$\frac{\pi}{\sqrt{C_2}} < \ell < \frac{\pi}{\sqrt{C_1}}$$

In particular if we set $\operatorname{diam}(M,g) := \sup_{x,y\in M} d(x,y)$, then

$$\operatorname{diam}(M,g) \leq \frac{\pi}{\sqrt{C_1}}.$$

Pf: Since $\sec_g \geq C_1 > 0$ by Rauch if $\tilde{\gamma} : [0,\ell] \to$ a normal geodesic in S^n_R where $R = 1/\sqrt{C_1}$ then since γ has no conj. points in $[0,\ell)$, so does $\tilde{\gamma}$ But we know $\tilde{\gamma}$ has to have a conj point at time $t = \pi R = \pi/\sqrt{C_1}$. So $\ell \leq \pi/\sqrt{C_1}$. Similarly the other direction.
For the diameter bound, if $x, y \in M$ & γ is a minimal geodesic connecting them $\gamma : [0,\ell] \to M$ has no conj. point. Again by Rauch's

$$d(x,y) = \ell < \pi/\sqrt{C_1}.$$

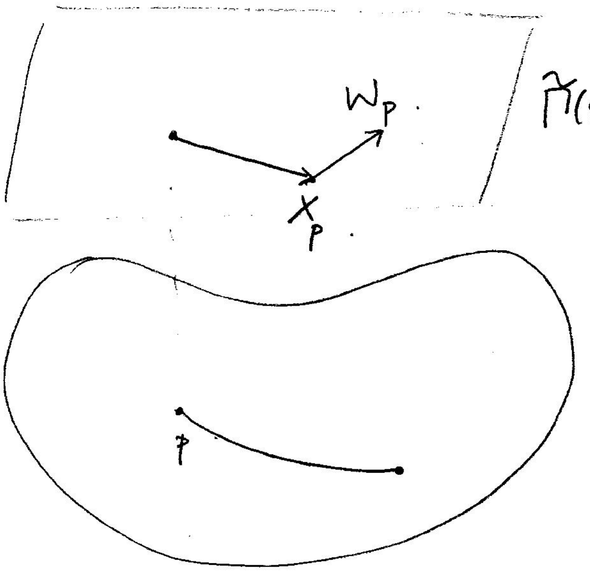
Cor 23.4: Let (M,g) be a complete, Riem. mfld w/ $\sec_g \leq 0$ (resp ≥ 0). Then for any $p \in M$ & any $X_p \in T_pM$, $W_p \in T_pM \cong T_{X_p}T_pM$,

$$|(d\exp_p)_{X_p}(W_p)| \geq |W_p| \quad (\text{resp. } \leq).$$

In particular, for any curve $\tilde{\gamma}$ in T_pM,

$$L(\tilde{\gamma}) \leq L(\exp_p \circ \tilde{\gamma}) \quad (\text{resp } \geq).$$

Pf:



$$\tilde{\Gamma}(s,t) = \tilde{\gamma}_s(t) = t\,(X_p + s\,W_p)$$

$$\downarrow$$

$$\gamma_s(t) = \exp_p \tilde{\gamma}_s(t).$$

$$\tilde{J} = \partial_s \tilde{\Gamma}(0,t) = t W_p, \quad J = \partial_s \Gamma(0,t) = t(d\exp_p)_{tX_p}(W_p).$$

Applying Rauch at $t=1$ to $(\tilde{M}, \tilde{g}) = (T_pM, g_p)$, since $\sec_{\tilde{g}} = 0$,

$$|W_p| \leq |(d\exp_p)_{X_p}(W_p)|.$$

<u>Cor 23.5</u>: Let (M,g) be complete, simply connected w/ $\sec_g \leq 0$. Consider a geodesic $\triangle ABC$ (i.e each side is a min. geod.) w/ side lengths a, b, c & opp angles α, β, γ. Then

(1) $a^2 + b^2 - 2ab\cos\gamma \leq c^2$.

(2) $\alpha + \beta + \gamma \leq \pi$.

<u>Pf</u> 1). In T_pM consider $\triangle OPQ$ where O is the origin, $|OP| = a$, $|OQ| = b$, $\angle O = \gamma$.

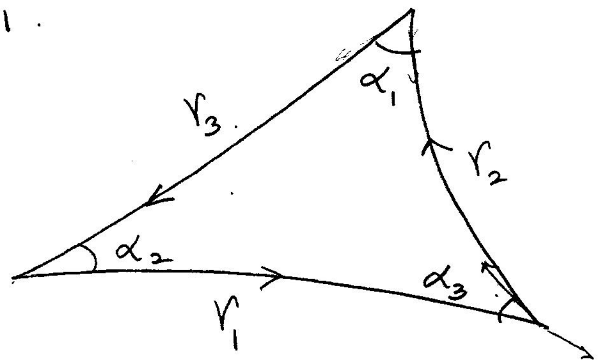
Let $\eta = \exp_p^{-1}(AB)$. Note by Cartan-Hadamard $\exp_p^{-1}$ is a diffeo.

Now $|PQ| \leq L(\eta) \leq c$. Now result follows from Euclidean cosine rule.

2). By (1) if $A'B'C'$ is a corresponding triangle in $\mathbb{R}^2$ w/ $|A'B'| = c$, $|B'C'| = a$, $|A'C'| = b$, then $\gamma \leq \gamma'$. Similarly $\alpha \leq \alpha'$, $\beta \leq \beta'$.

- TOPONOGOV'S THEOREM.

Def^n: A (min) geodesic Δ in (M, g) is a set of three (minimal) unit speed geodesics $\gamma_i : [0, l_i] \to M$ s.t $\gamma_i(l_i) = \gamma_{i+1}(0)$ (using cyclical order) w/ $l_{i+2} \leq l_i + l_{i+1}$.



We set α_i to be the angle between $-\gamma_{i+1}'(l_{i+1})$ and $\gamma_{i+2}'(0)$.

Note: Δ ineq $\Rightarrow$ $l_{i+2} \leq l_i + l_{i+1}$ if Δ is min-geodesic Δ

Th^m 23.6 Let (M, g) be a complete manifold satisfying $\sec_g \geq K$ (resp. $\leq K$). Let ABC be a geodesic Δ w/ AB minimal. Let $\tilde{A}\tilde{B}\tilde{C}$ be a comparison triangle in the model sp $(\tilde{M}_K, \tilde{g}_K)$ s.t $|\tilde{A}\tilde{B}| = |AB|$, $|\tilde{A}\tilde{C}| = |AC|$ & $\angle\tilde{A} = \angle A$.

Then

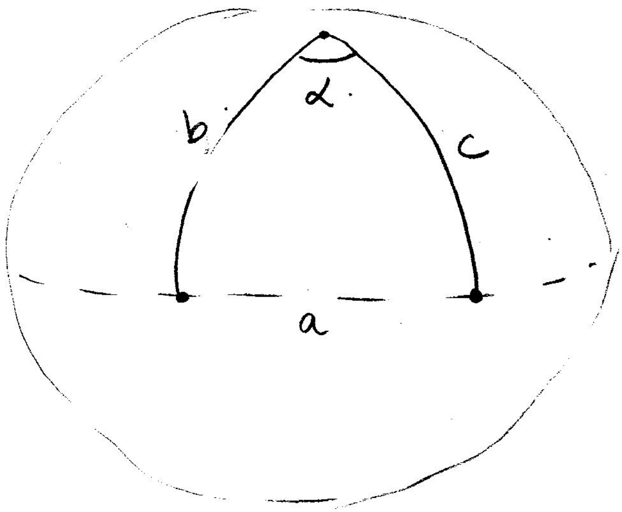
$$|BC| \leq |\tilde{B}\tilde{C}| \quad (\text{resp. } \geq).$$

<u>Cor 23.7</u>: Let (M,g) be a complete Riem. mfld. & let ABC be a min. geod. triangle w/ sides a, b, c & opp angles α, β, γ. If $\sec_g \geq 0$ (resp. ≤ 0) then

(1) $b^2 + c^2 - 2bc\cos\alpha \geq a^2$ (resp. $\leq$).

(2) $\alpha + \beta + \gamma \geq \pi$ (resp. $\leq$).

<u>Ex</u>: Consider the foll Δ on the sphere S^2.



$b = c = \pi/2$.

$a = \alpha$.

Then

$2b^2(1 - \cos\alpha) \geq \alpha^2$.