Lecture - 24

• Myers' Theorem

<u>Def^n</u>: We say that the Ricci curvature of (M,g) is <u>bounded below</u> (resp above) by λ, and write $Rc \geq \lambda$ (resp $Rc \leq \lambda$) if $\forall\ \xi \in TM$,

$$Rc(\xi, \xi) \geq \lambda |\xi|^2 \quad (\text{resp. } Rc(\xi,\xi) \leq \lambda |\xi$$

We say Ricci curvature is bounded by $\lambda > 0$ if $-\lambda \leq Rc \leq \lambda$

<u>Rk</u>: 1). $Rc \geq \lambda \iff Rc - \lambda g$ is a +ve definite symmetric $(0,2)$ tensor. So we sometimes write $Rc \geq \lambda g$.

2). $\sec_g \geq K \implies Rc \geq (n-1)K$. To see this, if $\{e_i\}$ is an o.n.b for T_pM, then

$$Rc(e_i, e_i) = \sum_{j \neq i} Rm(e_j, e_i, e_i, e_j)$$

$$= \sum_{j \neq i} \sec(e_i \wedge e_j)$$

$$\geq K \sum_{j \neq i} 1 = K(n-1)$$

The converse is not true if $n > 3$. For instance consider $S^2_{(1)} \times S^2_{(2)}$ w/ product of round metrics $g = g_{S^2_{(1)}} \oplus g_{S^2_{(2)}}$. If we normalize s.t $Rc(g_{S^2_{(1)}}) = g_{S^2_{(1)}}$, then $Rc(g) = g$. In particular $Rc\, g \geq 1$. But $\exists$ 2-plane Π s.t $K(\Pi) = 0$.

3) $-\lambda \le Rc \le \lambda \implies |Rc| \le \sqrt{n}\lambda$. Here recall that $|Rc|^2 = g^{ij} g^{k\ell} R_{ik} R_{j\ell}$. To see this, since Rc is symmetric & g is positive definite $\exists$ coordinates s.t at $p \in M$, $g_{ij} = \delta_{ij}$, & $R_{ij} = \lambda_i \delta_{ij}$. Then

$-\lambda \le Rc \le \lambda$ at $p \iff -\lambda \le \lambda_i \le \lambda$ $\forall$

$\iff |\lambda_i| \le \lambda \ \forall i \implies \left(\sum \lambda_i^2\right)^{1/2} \le \sqrt{n}\lambda$.

Conversely if $|Rc| \le \sqrt{n}\lambda$, then $|\lambda_i| \le \sqrt{n}\lambda \ \forall i$

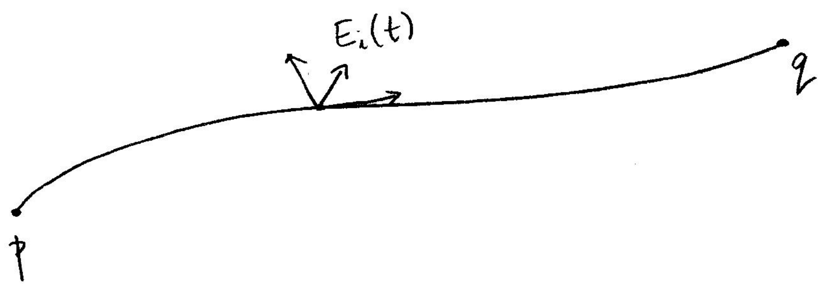
$\implies -\sqrt{n}\lambda \le Rc \le \sqrt{n}\lambda$. So boundedness of Ricci $\iff |Rc| \le \tilde{\lambda}$ for some $\tilde{\lambda}$.

<u>Th^m 24.1</u>: (Myers') Sps (M, g) is complete w/ $Rc \ge (n-1)\lambda$. Then

$$\operatorname{diam}(M, g) \le \frac{\pi}{\sqrt{\lambda}}.$$

In particular, M is compact & has finite fundamental group.

<u>Pf</u>: Let γ be a minimizing unit speed geodesic segment w/ $\mathcal{L}(\gamma) = L$.



Let $\{E_i(t)\}$ be a $\parallel$-o.n.b for $T_{\gamma(t)}M$ s.t $E_n(t) := \dot{\gamma}(t)$

For $i = 1, \cdots, n-1$, let

$$V_i(t) = \sin\frac{\pi t}{L} \cdot E_i(t)$$

(Rk: On S^n_R, these are basically the normal Jacobi fields).

Then $V_i(0) = V_i(L) = 0$.

$$D_t V_i = \frac{\pi}{L}\cos\left(\frac{\pi t}{L}\right) \cdot E_i(t)$$

$$D_t^2 V_i = -\frac{\pi^2}{L^2}\sin\left(\frac{\pi t}{L}\right) \cdot E_i(t).$$

So,

$$I_\gamma(V_i, V_i) = \int_0^L \langle V_i', V_i'\rangle - Rm(V_i, \dot{\gamma}, \dot{\gamma}, V_i)$$

$$= \int_0^L \frac{d}{dt}\langle V_i', V_i\rangle - \int_0^L \left(\langle V_i'', V_i\rangle + Rm(V_i, \dot{\gamma}, \dot{\gamma}, V_i)\right)$$

$$= \int_0^L \sin^2\left(\frac{\pi t}{L}\right)\left[\frac{\pi^2}{L^2} - Rm(E_i, \dot{\gamma}, \dot{\gamma}, E_i)\right].$$

Summing from $i=1$ to $n-1$, since E_i is o.n.b.

$$\sum_{i=1}^{n-1} I_\gamma(V_i, V_i) = \int_0^L \sin^2\left(\frac{\pi t}{L}\right)\left[\frac{(n-1)\pi^2}{L^2} - Rc(\dot{\gamma}, \dot{\gamma})\right].$$

$$\leq (n-1)\left(\frac{\pi^2}{L^2} - \lambda\right)\int_0^L \sin^2\left(\frac{\pi t}{L}\right)$$

γ minimizing $\Longrightarrow$ $I_\gamma(V_i, V_i) \geq 0 \quad \forall i$.

So

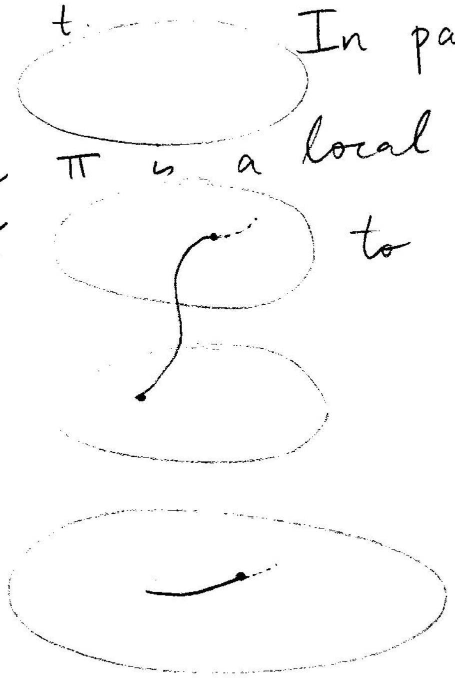
$$0 \leq (n-1)\left(\frac{\pi^2}{L^2} - \lambda\right) \int_0^L \sin^2\left(\frac{\pi t}{L}\right)$$

So $\pi^2/L^2 - \lambda \geq 0$ or $L \leq \pi/\sqrt{\lambda}$. Since this is true for any min. geodesic, diam $\leq \pi/\sqrt{\lambda}$.

Hopf - Rinow $\Longrightarrow$ M is compact.

<u>Claim</u> $\pi_1(M)$ is finite.

<u>Pf</u>: Let $\pi : \tilde{M} \longrightarrow M$ be the universal cover. Let $\tilde{g} = \pi^* g$. We claim that $(\tilde{M}, \tilde{g})$ is complete. If not, then $\exists$ a geodesic $\tilde{\gamma} : [0, a)$ which cannot be extended. Then $\gamma = \pi \circ \tilde{\gamma}$ is a geodesic on M & hence exists $\forall$ t. In particular exists on $[0, a+\varepsilon)$. Since π is a local diffeo, one can then extend $\tilde{\gamma}$ to $[0, a+\varepsilon)$. Contradiction!



M.

By Myer's, $\tilde{M}$ is compact. But $\pi_1(M) \cong \pi^{-1}(p)$ for each p. $\pi^{-1}(p)$ is a discrete set & hence has to be finite.

• Bishop – Gromov Volume Comparison

Recall: If (M^n, g) is an oriented Riem. mfld, then there is a natural volume form. If $g = g_{ij}\, dx^i \otimes dx^j$ Then

$$dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n .$$

where $dx^1, \ldots, dx^n$ is an oriented basis of T^*M. For any open set $U \subset M$, we denote the volume by

$$|U| = \int_U dV_g .$$

Th^m 24.2 Let (M^n, g) be a complete oriented mfld w/ $\mathrm{Ric} \geq (n-1)K$. Let $(\tilde{M}_K, \tilde{g}_K)$ be the model s.c space form w/ $\mathrm{Rc}\, g_k = (n-1)K$. For $p \in M$ & $\tilde{p} \in M_K$ let $B(p, r)$ & $B_K(\tilde{p}, r)$ denote balls of radius r in the respective metrics Then the function

$$r \longrightarrow \frac{|B(p, r)|}{|B_K(\tilde{p}, r)|}$$

is decreasing. In particular $|B(p, r)| \leq |B_K(\tilde{p}, r)|$. Moreover, we have equality if & only if $B(p, r)$ is isometric to $B_K(p, r)$.

• Discussion: Recall that if $\widetilde{Cut(p)}$ is the cut locus in the tangent sp, then if $\tilde{U}_p = T_pM \setminus \widetilde{Cut(p)}$

$$\exp_p : \tilde{U}_p \longrightarrow U_p = M \setminus Cut(p).$$

is a diffeo. Moreover, since $|Cut(p)| = 0$, for any open $\mathcal{U} \subset M$, letting $\tilde{\mathcal{U}} = \exp^{-1}(\mathcal{U} \cap \mathcal{U}_p)$.

$$|\mathcal{U}| = \int_{\tilde{\mathcal{U}}} \exp_p^* dV_g.$$

We can compute $\exp_p^* dV_g$ using Jacobi fields. We first need the foll "global" Gauss Lemma.

<u>Lemma 24.3</u>: Let $v \in T_pM$, $w \in T_v T_p M \cong T_pM$, and let $\gamma(t) = \exp_p(tv)$. Then

$$\langle \dot{\gamma}(t), (d\exp_p)_{tv}(w) \rangle = \langle v, w \rangle.$$

In particular, $\gamma(t)$ is $\perp$ to the sphere

$$S_r = \{q \in M \mid d(p,q) = r\}.$$

<u>Pf</u>: Let $J(t) = (d\exp_p)_{tv}(tw)$. Then J is a Jacobi field along γ w/

$$J(0) = 0, \quad J'(0) = w$$

We know that $\langle \dot{\gamma}, J \rangle$ is a linear function. In fact

$$\langle \dot{\gamma}, J \rangle = \langle \dot{\gamma}(0), J(0) \rangle + t \langle \dot{\gamma}(0), J'(0) \rangle$$

$$= t \langle v, w \rangle$$

So $\langle \dot{\gamma}, (d\exp_p)_{tv}(w) \rangle = \langle v, w \rangle.$

Continuing w/ our discussion we work w/ normal coordinates & the corresponding polar coordinates on $\tilde{\mathcal{U}}_p$. Fix a direction $\vec{\theta} \in S_pM$. Let $\vec{\theta}, e_2, \dots, e_n$ be an o.n.f for T_pM & let $(x^1, \dots, x^n)$ be the corresponding normal coordinates. Note, that the Lebesgue meas is

$$dx^1 \wedge \dots \wedge dx^n = r^{n-1} dr \wedge d\sigma_{n-1} \leftarrow \text{st - meas on } S^{n-1}$$

We can the write

$$\exp_p^* dV_g = \mathcal{A}(r,\theta) \cdot r^{n-1} dr \wedge d\sigma_{n-1}$$

where $\mathcal{A}(r,\theta) = \sqrt{\det g_{ij}(\exp_p(r\vec{\theta})}$

Let J_i be ! Jacobi-field s.t $J_i(0,\vec{\theta}) = 0$ & $J_i'(0,\vec{\theta}) = e_i$. Note that $J_i \in \mathcal{J}^\perp(\exp_p t\vec{\theta})$. Also

$$\boxed{J_i(t,\vec{\theta}) = t(d\exp_p)_{t\vec{\theta}}(e_i)}$$

<u>Claim</u>: $\mathcal{A}(r,\theta) = r^{-n-1} \left| \frac{\partial}{\partial r} \wedge J_2 \wedge \dots \wedge J_n \right|$

<u>Pf</u>: Under $\exp_p|_{\tilde{\mathcal{U}}_p}$, $(x^1, \dots, x^n)$ give coordinates near $\exp_p(r\vec{\theta})$. In fact

$$\partial_1\Big|_{\exp_p(r\vec{\theta})} = \dot{\gamma}(r) = \frac{\partial}{\partial r}, \quad \partial_i\Big|_{\exp_p(r\vec{\theta})} = \frac{J_i}{r}, \quad i = 2, \dots, n$$

If we denote by $e_i(t)$, the $\|$-transp. of e_i along $\gamma(t) = \exp_p(t\vec{\theta})$, for $i = 2, \ldots, n$, then since by Gauss' lemma, $J_i \perp \partial/\partial r$, we have

$$J_i = \sum_{k=2}^{n} a_{ik} e_k \quad , \quad A = \{a_{ik}\} \, ; \; D = AA^T = (d_{ij}).$$

Also $\{e_k\}$ are o.n. So for $i, j \geq 2$

$$g_{ij} := g(\partial_i, \partial_j) = r^{-2} g(J_i, J_j)$$

$$= r^{-2} \sum_k a_{ik} a_{jk} = r^{-2} d_{ij}.$$

$$g_{11} = 1, \quad g_{1i} = 0 \quad \forall \, i \geq 2.$$

So $\sqrt{\det g_{ij}} = r^{-(n-1)} \sqrt{\det(AA^T)} = r^{-(n-1)} |\det(A)|$.

But $J_2 \wedge \cdots \wedge J_n = \det(A) \, e_2 \wedge \cdots e_n$.

So $|J_2 \wedge \cdots \wedge J_n| = |\det A|$

So

$$\mathcal{A}(r, \vec{\theta}) = r^{-(n-1)} \left| \frac{\partial}{\partial r} \wedge J_2 \wedge \cdots J_n \right|$$

<u>Example</u>: Let $(M, g) = (S^n, g_{S^n})$, $p = (0, 0, \ldots, 1)$.

Let $\gamma(t)$ be a great circle & $\{E_i(t)\}_{i=2}^{n}$ be a $\|$-o.n.f along $\gamma(t)$. Let J_i be the Jacobi field w/ $J_i(0) = 0$, $J_i'(0) = E_i(0)$.

Then by our earlier work $J_i(t) = \sin(t)\, E_i(t)$

E_i o.n. $\Longrightarrow$

$$A(r, \theta) = r^{-(n-1)} \sin^{n-1} r.$$

In particular.

$$|S^n| = \int_{S^{n-1}} \int_0^{\pi} r^{-(n-1)} \sin^{n-1} r \cdot r^{n-1}\, dr\, d\sigma_n.$$

$$= |S^{n-1}| \int_0^{\pi} \sin^{n-1} r\, dr$$

$$= 2\pi^{\frac{n+1}{2}} / \Gamma\left(\frac{n+1}{2}\right).$$