LECTURE - 25

• Recall: $\exp_p : \tilde{\mathcal{U}}_p \longrightarrow \mathcal{U}_p$ is a diffeo. Fix a direction $\vec{\theta} \in S_pM$, and a orthonormal, oriented basis $\{\vec{\theta}, e_2, \ldots, e_n\}$ for T_pM w/ corresponding normal coordinates $\{x^1, \ldots, x^n\}$ & polar coordinates $(r, \vec{\theta})$. Then

$$\exp_p^* dV_g = \mathcal{A}(r, \vec{\theta})\, r^{n-1}\, dr \wedge d\sigma_{n-1}, \text{ where}$$

$$\mathcal{A}(r, \vec{\theta}) = r^{-(n-1)} \left| \partial_r \wedge J_2 \wedge \cdots \wedge J_n \right|$$

$$J_i(r, \vec{\theta}) = t\left(d\exp_p\right)_{t\vec{\theta}}(e_i)$$

Thm (Bishop - Gromov) Let (M^n, g) be complete oriented w/ $\mathrm{Ric} \geq (n-1)K$. Let $(\tilde{M}^n_K, \tilde{g}_K)$ be the s.c sp. form w/ $\sec_{\tilde{g}_K} \equiv K$. For $p, \tilde{p} \in M, \tilde{M}_K$, let $B(p, r)$ & $B_K(\tilde{p}, r)$ denote the respective metric balls of radius r. Then

$$r \longrightarrow \frac{|B(p, r)|}{|B_K(p, r)|}$$

is decreasing. In particular, $|B(p, r)| \leq |B_K(\tilde{p}, r)|$ w/ equality $\Longleftrightarrow$ $B(p, r) \cong_{\text{isom}} B_K(\tilde{p}, r)$.

• For the model sp., we have $dV_g = \mathcal{A}_K(r, \vec{\theta})\, r^{n-1} dr \wedge d\sigma_{n-1}$ with $\mathcal{A}_K(r, \vec{\theta}) = \mathcal{A}_K(r) = \left(\dfrac{\mathrm{sn}_K(t)}{r}\right)^n$ where

②

$$\mathrm{sn}_K(t) = \begin{cases} \dfrac{\sin(\sqrt{K}\cdot t)}{\sqrt{K}} & , K > 0 \\ t & K = 0 \\ \dfrac{\sinh(\sqrt{-K}\,t)}{\sqrt{-K}} & , K < 0 \end{cases}$$

Lemma 25.1 If $\mathrm{Ric}_g \geq (n-1)K$, then $\forall\, r < \overset{\text{cut-time}}{\rho(\vec{\theta})}$

$$\frac{d}{dr}\log A(r,\vec{\theta}) \leq \frac{d}{dr}\log A_K(r).$$

In particular $A(r,\vec{\theta}) \leq A_K(r)$ w/ eq $\Leftrightarrow$ $B(p,r) \cong B_K(\tilde{p}, r)$

Pf of Bishop-Gromov Define

$$a(r) = \int_{S^{n-1}} A(r,\vec{\theta})\, d\sigma_{n-1}(\vec{\theta}), \quad a_K(r) = \int_{S^{n-1}} A_K(r)\, d\sigma_{n-1}$$

$$= A_K(r)\,|S^{n-1}|.$$

$$V(r) = |B(p,r)| / |B_K(\tilde{p}, r)|.$$

Then

$$\frac{d}{dr}\log V(r) = \frac{d}{dr}\log\int_0^r a(t)\,dt - \frac{d}{dr}\log\int_0^r a_K(t)\,dt$$

$$= \frac{a(r)}{\int_0^r a(t)\,dt} - \frac{a_K(r)}{\int_0^r a_K(t)\,dt}$$

$$= \frac{\int_0^r [a_K(t)\,a(r) - a_K(r)\,a(t)]\,dt}{\left(\int_0^r a(t)\,dt\right)\left(\int_0^r a_K(t)\,dt\right)}$$

To complete the proof that $V(r)\downarrow$, we need.

<u>Claim</u> $a_K(r)\,a(t) - a_K(t)\,a(r) \geqslant 0$ if $t \leq r$

<u>Pf</u>: Note that if $t \leq r$.

$$\frac{a(t)}{a_K(t)} = \frac{1}{|S^{n-1}|}\int \frac{\mathcal{A}(t,\vec{\theta})}{\mathcal{A}_K(t)}\,d\sigma_{n-1}(\vec{\theta}).$$

$$\geqslant \frac{1}{|S^{n-1}|}\int \frac{\mathcal{A}(r,\vec{\theta})}{\mathcal{A}_K(r)}\,d\sigma_{n-1}(\vec{\theta})$$

$$= \frac{a(r)}{a_K(r)}$$

Next, since $\lim_{r\to 0^+} \frac{|B(p,r)|}{|B_K(\tilde{p},r)|} = 1$, we see that

$V(r) \leq 1 \;\; \forall\, r$ i.e. $|B(p,r)| \leq |B_K(\tilde{p},r)|$.

In fact this follows directly from Lemma 25.1 since $a(r) \leq a_K(r)$. Also, again by Lemma, $a(r) = a_K(r) \iff \mathcal{A}(r,\vec{\theta}) = \mathcal{A}_K(r) \iff B(p,r) \cong B_K(\tilde{p},r)$.

<u>Pf of Lemma 25.1</u>: As before fix $\vec{\theta} \in S_pM$, and an o.n.f $\{\vec{\theta}, e_2, \ldots, e_n\}$ for T_pM. Let $\gamma(t) = \exp_p(t\vec{\theta})$ & extend this to a ∥-o.n.f $\{\frac{\partial}{\partial r}, e_2(t), \ldots, e_n(t)\}$ along $\gamma(t)$. Recall that

$$J_i(t) = t(d\exp)_{t\vec{\theta}}(e_i), \quad J_i(0) = 0, \quad J_i'(0) = e_i(0).$$

For any $r < \rho(\vec{\theta})$, and $i \geqslant 2$, let $J_i^r(r) = e_i(r)$, and let $J_i^r(t)$ be the unique Jacobi field s.t $\quad J_i^r(0) = 0, \quad J_i^r(r) = e_i(r)$.

<u>Claim</u>: $\mathcal{A}(t, \vec{\theta}) = C_r t^{1-n} \, |\dot{\gamma}(t) \wedge J_2^r(t) \wedge \cdots \wedge J_n^r(t)|$

where $C_r^{-1} = |\dot{\gamma}(t) \wedge J_2^{r\prime}(0) \wedge \cdots \wedge J_n^{r\prime}(0)|$

<u>Pf</u>: Sps $J_i(t) = \sum a_{ij}(t) e_j(t)$, $A(t) = (a_{ij}(t))$ & $B(t) = (b_{ij}(t)) = A(t)^{-1}$. Then $\forall i$ $\sum b_{ik}(r) J_k(t)$ is a Jacobi field vanishing at $t=0$ & $e_i(r)$ at $t=r$. So uniqueness $\Longrightarrow$

$$J_i^r(t) = \sum_{k=2}^{n} b_{ik}(r) J_k(t)$$

So $$\mathcal{A}(t, \vec{\theta}) = t^{1-n} \, |\dot{\gamma}(t) \wedge J_2(t) \wedge \cdots \wedge J_n(t)|$$

$$= \frac{t^{1-n} \, |\dot{\gamma}(t) \wedge J_2^r(t) \wedge \cdots \wedge J_n^r(t)|}{\det B(r)}$$

$$= C_r t^{1-n} \, |\dot{\gamma}(t) \wedge \cdots \wedge J_n^r(t)|$$

where $C_r^{-1} = \det B(r)$

But notice $$(J_i^r)'(0) = \sum_{k=2}^{n} b_{ik}(r) J_k'(0)$$

$$= \sum_{k=2}^{n} b_{ik}(r) e_k(0).$$

Since $\{e_k(0)\}$ is o.n., $\det B(r) = |(J_2^r)'(0) \wedge \cdots (J_n^r)'(0)|$

Note that, since $J_i^r \perp \dot{\gamma}(t)$,

$$|\dot{\gamma}(t) \wedge J_2^r(t) \wedge \cdots \wedge J_n^r(t)| = \det\left(\langle J_i^r(t), J_k^r(t)\rangle\right)^{1/2}$$

$$= D(t)^{1/2}.$$

Then Claim $\Longrightarrow$

$$\frac{A'(t,\vec{\theta})}{A(t,\vec{\theta})} = \frac{D'(t)}{2D(t)} - \frac{n-1}{t} \qquad (*)$$

If $P(t) = (P_{ik}(t)) = (\langle J_i^r(t), J_k^r(t)\rangle)$, then

$$D'(t) = D(t)\, Tr\,(P(t)^{-1} P'(t))$$

But note that $P(r) = (\langle e_i(r), e_k(r)\rangle) = id.$

So

$$\frac{D'(r)}{D(r)} = Tr\, P'(r).$$

$$= 2\sum_{i=2}^{n} \langle (J_i^r)'(r), J_i^r(r)\rangle$$

So plugging in $t = r$ in $(*)$

$$\frac{A'(r,\vec{\theta})}{A(r,\vec{\theta})} = \sum_{i=2}^{n} \langle J_i^{r\prime}(r), J_i^r(r)\rangle - \frac{n-1}{r} \qquad (**)$$

By the index formula

$$I_r(J_i^r, J_i^r) = \int_0^r \langle J_i^{r\prime}, J_i^{r\prime}\rangle - Rm(J_i^r, \dot{\gamma}, \dot{\gamma}, J_i^r)$$

$$= \langle (J_i^r)', J_i^r\rangle \Big|_{t=0}^{r} \qquad \text{(integration by parts \& } J_i^r \text{ is Jacobi)}$$

$$= \langle (J_i^r)'(r), J_i^r(r)\rangle.$$

So
$$\frac{A'(r,\vec\theta)}{A(r,\vec\theta)} = \sum_{i=2}^{n} I_r(J_i^r, J_i^r) - \frac{n-1}{r}. \quad (***)$$

Now, consider the vector field (for $i \geq 2$)

$$X_i^r(t) = \frac{sn_K(t)}{sn_K(r)} e_i(t).$$

Then $X_i^r(0) = 0 = J_i^r(0)$, $X_i^r(r) = e_i(r) = J_i^r(r)$.

So $I_r(J_i^r, J_i^r) \leq I_r(X_i^r, X_i^r)$ and hence

$$\frac{A'(r,\vec\theta)}{A(r,\vec\theta)} + \frac{n-1}{r} \leq \sum_{i=2}^{n} I_r(X_i^r, X_i^r)$$

Recalling that $sn_K''(t) + K\, sn_K(t) = 0$, we see that

$$D_t^2 X_i^r(t) = -K \frac{sn_K(t)}{sn_K(r)} e_i(t)$$

So

$$I_r(X_i^r, X_i^r) = \langle D_t X_i^r(r), X_i^r(r)\rangle + \int \left(\frac{sn_K(t)}{sn_K(r)}\right)^2 (K - Rm(e_i, \dot r, \dot r, e_i))$$

Summing up.

$$\frac{A'(r,\vec\theta)}{A(r,\vec\theta)} + \frac{n-1}{r} \leq \sum_{i=2}^{n} \langle D_t X_i^r(r), X_i^r(r)\rangle$$

$$= \frac{sn_K'(r)}{sn_K(r)} (n-1)$$

On the other hand, for the model space, one can simply take $\tilde{J}_i(t) = sn_K(t)\, e_i(t)$.

Then

$$\mathcal{A}_K(t, \vec{\theta}) = \left(sn_K(t)\right)^{n-1} t^{1-n}.$$

So
$$\frac{\mathcal{A}_K'(r)}{\mathcal{A}_K(r)} + \frac{n-1}{r} = \frac{sn_K'(r)}{sn_K(r)}(n-1).$$

So
$$\frac{d \ln \mathcal{A}(r, \vec{\theta})}{dr} = \frac{\mathcal{A}'(r, \theta)}{\mathcal{A}(r, \theta)} \leq \frac{\mathcal{A}_K'(r)}{\mathcal{A}_K(r)} = \frac{d}{dr} \ln \mathcal{A}_K(r)$$

We have equality $\iff$ $J_i^r = X_i^r$ $\forall$ i (and $\forall\, \theta$).

<u>Exercise</u>: Prove that this implies that $B(p, r)$ $\cong B_K(\tilde{p}, r)$. (Hint: Refer to our proof that all space forms are locally isometric).

Also clearly $\displaystyle\lim_{r \to 0^+} \frac{\mathcal{A}(r, \vec{\theta})}{\mathcal{A}_K(r)} = 1$.

So $\mathcal{A}(r, \vec{\theta}) \leq \mathcal{A}_K(r)$ $\forall$ r.

- <u>GEOMETRIC INTERPRETATION OF $\mathcal{A}'(r, \vec{\theta}) / \mathcal{A}(r, \vec{\theta})$</u>

Fix $p \in M$, and consider $\rho(x) := d(p, x)$. If $x \notin \mathrm{Cut}(p)$, then $|\nabla \rho| = 1$, so $S_r := \{\rho(x) = r\}$ is a smooth mfld. We let $m(r, \vec{\theta})$ be the <u>mean curvature</u> of $S_r \subsetneq \mathcal{U}_p$ at $x = \exp_p(r\vec{\theta})$ w.r.t $-\partial_r$

<u>Prop 25.2</u> At any $x \in \mathcal{U}_p$ w/ $x = \exp_p(r\vec{\theta})$.

$$\frac{\mathcal{A}'(r, \vec{\theta})}{\mathcal{A}(r, \vec{\theta})} + \frac{n-1}{r} = \Delta \rho = m(r, \vec{\theta}).$$

Pf: By an assignment problem, the scalar 2nd fund. form of $\rho = r$ is given (w.r.t $\vec{n} = -\nabla\rho$) by

$$h(V, W) = \frac{\text{Hess}\,\rho(V, W)}{|\nabla\rho|}$$

So $m(r, \vec{\theta}) = \operatorname{tr} \text{Hess}\,\rho = \Delta\rho$.

We now compute $m(r, \vec{\theta})$ in terms of the Jacobi fields J_i^r. Recall that $J_i^r(r) = e_i(r)$.

By Weingarten w/ $N = -\partial_r = -\dot{r}(r)$

$$\begin{aligned} m(r; \vec{\theta}) &= \sum_{i=2}^{n} h(e_i(r), e_i(r)) \\ &= \sum_{i=2}^{n} \langle \nabla_{e_i(r)} \dot{r}(r), e_i(r) \rangle \\ &= \sum_{i=2}^{n} \langle (J_i^r)'(r), J_i^r(r) \rangle \\ &= \frac{A'(r, \vec{\theta})}{A(r, \vec{\theta})} + \frac{n-1}{r} \quad \text{by } (**) \end{aligned}$$

Then Lemma 25.1 $\Longleftrightarrow$

Prop 25.3 (Laplace comparison) If $\text{Ric} \geq (n-1)K$, then $\forall\, r < \rho(\vec{\theta})$

$$\Delta\rho \leq \Delta_K \rho_K$$

Rk Bishop-Gromov follows directly from this

Cor 25.4 Let (M, g) be complete w/ $Ric \geq (n-1)K$

(a) If $K > 0$, then $\exists$ dimensional constant C_n

s.t

$$|B(p, r)| \geq C_n K^{n/2} r^n$$

(b) If $K = 0$, M is non-comp. then $\exists$ const. $C = C(p, n)$ s.t $\forall r > 2$

$$|B(p, r)| \geq C r$$

Rk: If $K = 0$, one cannot do better than linear volume growth. e.g cylinder.

• APPLICATION TO $\pi_1(M)$.

Def^n 1) Let G be a finitely generated group w/ generators $\Gamma = \{g_1, \ldots, g_N\}$. The growth function of G w.r.t Γ is defined by

$$N_G^\Gamma(k) = \# \{g \in G \mid \exists\, l \leq k \ \&\ g_{i_1}, \ldots, g_{i_l} \in \Gamma \text{ s.t } g = g_{i_1}^{\pm 1} \cdots g_{i_l}^{\pm 1}\}.$$

2) We say G has poly. growth if $\exists\, c > 0$ s.t

$$N_G^\Gamma(k) \leq c k^n.$$

Th^m (Milnor) Let M be complete w/ $Ric \geq 0$. & $G \subset \pi_1(M)$ be finetely gen. Then $\exists\, c = c(g, \Gamma)$ s.t

$$N_G^\Gamma(k) \leq c k^n$$