# LECTURE - 26.



## • BÖCHNER FORMULA FOR FUNCTIONS.

Lemma 26.1 For any $f \in C^\infty(M)$,

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla\nabla f|^2 + \langle \nabla \Delta f, \nabla f\rangle + Ric(\nabla f, \nabla f).$$

Pf (without normal coordinates). Recall $\Delta h = \mathrm{div}(\mathrm{grad}\, h) = \nabla_i \nabla^i h$.

$$|\nabla\nabla f|^2 = \nabla_i \nabla_j f \cdot \nabla_k \nabla_l f \cdot g^{ik} g^{jl} = \nabla_i \nabla_j f\, \nabla^i \nabla^j f.$$

We compute

$$\begin{aligned}
\Delta|\nabla f|^2 &= \nabla_i \nabla^i \nabla_j f \cdot \nabla^j f \\
&= \nabla_i \left(\nabla^i \nabla_j f\, \nabla^j f + \nabla_j f \cdot \nabla^i \nabla^j f\right) \\
&= \nabla^i \nabla_j f \cdot \nabla_i \nabla^j f + \nabla_i \nabla^i \nabla_j f\, \nabla^j f \\
&\quad + \nabla_i \nabla_j f\, \nabla^i \nabla^j f + \nabla_j f\, \nabla_i \nabla^i \nabla^j f
\end{aligned}$$

Now

$$\begin{aligned}
\nabla^i \nabla_j f\, \nabla_i \nabla^j f &= g^{ik} \nabla_k \nabla_j f \cdot \nabla_i \nabla^j f \\
&= \nabla_k \nabla_j f\, \nabla^k \nabla^j f = |\nabla\nabla f|^2
\end{aligned}$$

So,

$$\Delta|\nabla f|^2 = 2|\nabla\nabla f|^2 + \nabla_i \nabla^i \nabla_j f \cdot \nabla^j f + \nabla_j f\, \nabla_i \nabla^i \nabla^j f$$

Also,

$$\begin{aligned}
\nabla_i \nabla^i \nabla_j f &= \nabla_i \nabla_j \nabla^i f \\
&= \nabla_j \nabla_i \nabla^i f + R_{ijk}{}^{i} \nabla^k f.
\end{aligned}$$

$$= \nabla_j \Delta f + R_{kj} \nabla^k f$$

Similarly $\nabla_i \nabla^i \nabla^j f = \nabla_i \nabla^j \nabla^i f$

$$= g^{jk} \nabla_i \nabla_k \nabla^i f$$

$$= g^{jk} \left( \nabla_k \Delta f + R_{ikl}{}^{i} \nabla^l f \right)$$

$$= \nabla^j \Delta f + g^{jk} R_{kl} \nabla^l f$$

So

$$\Delta |\nabla f|^2 = 2 |\nabla \nabla f|^2 + 2 \langle \nabla \Delta f, \nabla f \rangle$$
$$+ 2 \mathrm{Ric}(\nabla f, \nabla f).$$

<u>$2^{nd}$ pf</u> (w/ normal coordinates) We compute at $p \in M$ & choose coordinates s.t $g_{ij}(p) = \delta_{ij}$ & $g_{ij;k}(p) = 0$

Here for any tensor, $T_{;k}(p) = \partial_k T(p)$

In normal coordinates

$$\Delta h(p) = \sum_i h_{;ii}(p)$$

$$(\mathrm{Hess}\, f)_{ij}(p) = f_{;ij}(p).$$

So $|\mathrm{Hess}\, f|^2(p) = \sum_{i,j} f_{;ij}^2$

Also the commutation identity at $p$ becomes

$$f_{;ikj} - f_{;ijk} = \nabla_j \nabla_k \nabla_i f - \nabla_k \nabla_j \nabla_i f$$

$$= g_{ip} (\nabla_j \nabla_k - \nabla_k \nabla_j)(\nabla^p f)$$

$$= g_{ip} R_{jkl}{}^{p} \nabla^l f.$$

$$= R_{jkli} f_{il}$$

So

$$\boxed{f_{iikj} - f_{iijk} = R_{jkli} f_{il}.}$$

We now compute

$$\Delta |\nabla f|^2 = \nabla_k \nabla^k (g^{ij} f_{;i} f_{;j})$$

$$= g^{ij} \nabla_k \nabla^k (f_{;i} f_{;j}).$$

$$= (f_{;j} f_{;j})_{kk}$$

$$= 2|\nabla\nabla f|^2 + 2 f_{;jkk} f_{;j}$$

$$= 2|\nabla\nabla f|^2 + 2 f_{;kjk} f_{;j}$$

$$= 2|\nabla\nabla f|^2 + 2 f_{;kkj} f_{;j} + 2 R_{kjlk} f_{;l} f_{;j}.$$

$$= 2|\nabla\nabla f|^2 + 2\langle \nabla \Delta f, \nabla f\rangle$$
$$+ 2 \mathrm{Ric}(\nabla f, \nabla f).$$

<u>Cor 26.2</u> Sps $(M, g)$ is compact w/ $\mathrm{Ric}\, g \geq t g$. If $\exists$ $u \in C^\infty(M)$, $u \not\equiv$ const & $\lambda \in \mathbb{R}$ s.t $-\Delta u = \lambda u$, then $\lambda \geq t$.

<u>Rk</u>: $\lambda \in \mathbb{R}$ is an eigenvalue of $-\Delta$ if $\exists$ $u \in C^\infty(M)$ s.t $u \not\equiv 0$ & $-\Delta u = \lambda u$. Multiplying by $u$ & integrating by parts if $u \not\equiv$ const.

$$\lambda \int u^2 = \int |\nabla u|^2 > 0$$

So $\lambda > 0$.

Pf: Let $\lambda > 0$ be an eigenvalue w/ corresponding eigenfunction $u \in C^\infty(M)$. Then

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla\nabla u|^2 - \lambda|\nabla u|^2 + \mathrm{Ric}(\nabla u, \nabla u)$$

$$> |\nabla\nabla u|^2 + (t-\lambda)|\nabla u|^2.$$

$$> (t-\lambda)|\nabla u|^2$$

Integrating $(t-\lambda)\int|\nabla u|^2 < 0$

or $t - \lambda < 0$ or $\lambda > t$

• APPLICATION TO LAPLACE COMPARISON

Prop 26.3 Let $(M, g)$ be a complete mfld w/ $\mathrm{Ric}_g \geq (n-1)K$. Fix $p \in M$, and let $\rho(x) := d(p, x)$. Sps $(\tilde{M}_K, \tilde{g}_K)$ is the model sp form w/ $\sec_{\tilde{g}_K} \equiv K$ & let $\tilde{p} \in \tilde{M}_K$ & $\tilde{\rho}(\tilde{x}) = d_{\tilde{g}_K}(\tilde{x}, \tilde{p})$. If $x \notin \mathrm{Cut}_g(p)$ & $\tilde{x} \notin \mathrm{Cut}_{\tilde{g}}(\tilde{p}) \subset \tilde{M}$ with $\rho(x) = \tilde{\rho}(x) = r$ (say) then

$$\Delta\rho(x) \leq \tilde{\Delta}\tilde{\rho} = (n-1)\frac{\mathrm{sn}_K'(r)}{\mathrm{sn}_K(r)}.$$

Pf: Since $x \notin \mathrm{Cut}_g(p)$, $\rho$ is smooth near $x$ w/ $|\nabla\rho|^2 \equiv 1$. So by Böchner formula.

$$|\nabla\nabla\rho|^2 + \langle\nabla\Delta\rho, \nabla\rho\rangle + \mathrm{Ric}(\nabla\rho, \nabla\rho) = 0.$$

<u>Claim 1</u> $|\nabla\nabla\rho|^2 \geq \dfrac{(\Delta\rho)^2}{n-1}$

<u>Pf</u>: Choose an o.n.f $(E_1, \dots, E_n)$ at $T_xM$ s.t. $E_n = \partial_r = \nabla\rho$. Then

$$|\nabla\nabla\rho|^2 = \sum_{i,j=1}^{n} \mathrm{Hess}\,\rho(E_i, E_j)^2$$

Note that $\mathrm{Hess}\,\rho(\partial_r, E_i) = g(\nabla_{E_i}\nabla\rho, \nabla\rho)$

$= 0$ since $|\nabla\rho|^2 \equiv 1$.

So $|\nabla\nabla\rho|^2 = \displaystyle\sum_{i,j=1}^{n-1} \mathrm{Hess}\,\rho(E_i, E_j)^2$

Let $a_{ij} = \mathrm{Hess}\,\rho(E_i, E_j)$. By Cauchy-Schwarz

$$(\Delta\rho)^2 = \left(\sum_{i=1}^{n-1} a_{ii}\right)^2 \leq \left(\sum_{i=1}^{n-1} a_{ii}^2\right)\left(\sum_{i=1}^{n-1} 1\right)$$

$$\leq \left(\sum_{i,j=1}^{n-1} a_{ij}^2\right)(n-1) = (n-1)|\nabla\nabla\rho|^2.$$

So, if $\mathrm{Ric} \geq (n-1)K$, then by claim,

$$\frac{(\Delta\rho)^2}{n-1} + \langle \nabla\Delta\rho, \nabla\rho \rangle + (n-1)K \leq 0 \quad (*)$$

Let $\gamma, \tilde{\gamma} : [0, r] \to M, \tilde{M}_K$ be unit speed minimal geodesics from $p, \tilde{p}$ to $x, \tilde{x}$ resp. Let $u(t) = \Delta\rho(\gamma(t))$ and $\tilde{u}(t) = \tilde{\Delta}\tilde{\rho}(\tilde{\gamma}(t))/_{n-1}$. Then $(*) \Longrightarrow$

$$\dot{u}(t) + u^2 \leq -K \quad (**)$$

Claim 2 $\tilde{\Delta}\tilde{\rho}(\tilde{r}(t)) = (n-1)\dfrac{sn_K'(t)}{sn_K(t)}$.

Pf: We saw last lecture that

$$\tilde{\Delta}\tilde{\rho}(\tilde{r}(t)) = \frac{d}{dt}\log A_K(t) + \frac{n-1}{t}.$$

$$= \frac{d}{dt}\log\left(\frac{sn_K(t)}{t}\right)^{n-1} + \frac{n-1}{t}.$$

$$= n-1\,\frac{d}{dt}\log sn_K(t)$$

$$= (n-1)\frac{sn_K'(t)}{sn_K(t)}.$$

Then if $\tilde{u}(t) = \tilde{\Delta}\tilde{\rho}(\tilde{r}(t))/(n-1)$, since $sn_K''(t) = -K\,sn_K(t)$

we have

$$\tilde{u}'(t) + \tilde{u}^2 = \left[\frac{sn_K(t)\,sn_K''(t) - sn_K'(t)^2}{sn_K(t)^2}\right].$$

$$= \left(\frac{sn_K'(t)}{sn_K(t)}\right)^2.$$

$$= -K. \qquad (***).$$

Hence we have

$$(u-\tilde{u})' \leq -(u^2-\tilde{u}^2). \qquad (\#).$$

Now define $f(t) = sn_K^2(u-\tilde{u})$. Then.

$$f'(t) = 2\,sn_K(t)\,sn_K'(t)(u-\tilde{u}) + sn_K^2(u-\tilde{u})'$$

$$= 2\,sn_K^2\,\tilde{u}(u-\tilde{u}) + sn_K^2(u-\tilde{u})'$$

$$\leq 2 sn_K^2 \tilde{u}(u-\tilde{u}) - sn_K^2(u^2 - \tilde{u}^2)$$

$$= -sn_K^2 (u^2 - \tilde{u}^2 - 2u\tilde{u} + 2\tilde{u}^2)$$

$$= -sn_K^2 (u-\tilde{u})^2 \leq 0 .$$

So $f$ is decreasing

Claim 3 $\quad \lim_{t\to 0} u(t) - \tilde{u}(t) = 0$

Assuming this, then since $\lim_{t\to 0} sn_K^2(t) = 0$ we have $\lim_{t\to 0} f(t) = 0$, and so $f(t) \leq 0 \;\forall\; t$

Hence $\quad u(t) \leq \tilde{u}(t) \;\forall\; t$ & in particular

$$\Delta \rho(x) \leq \tilde{\Delta}\tilde{\rho}(\tilde{x}).$$

Pf of Claim 3 $\quad$ We compute in normal coordinates. Fix a direction $\vec{\theta}$ & extend it to an o.n.f. $\{\vec{\theta}, \vec{e}_2, \ldots, \vec{e}_n\}$ of $T_pM$. Let $J_i$ ! Jacobi field s.t. $J_i(0) = 0$, $J_i'(0) = e_i$. If $\{x^i\}$ is the corresponding normal coordinates, then $\vec{\theta} = \partial_1|_p, \ldots$ $e_n = \partial_n|_p$. Also $J_i(t) = t \cdot \partial_i\big|_{(t,0,0,\ldots,0)}$. Now

$$\mathcal{A}(t,\vec{\theta}) = t^{-(n-1)} |\vec{\theta} \wedge \cdots \wedge J_n| .$$

$$= |\partial_1 \wedge \partial_2 \wedge \cdots \wedge \partial_n|_{g_{ij}(t,0,\ldots,0)} .$$

Gauss Lemma $\Rightarrow \partial_1 \perp \partial_i \;\forall\; i \geq 2 .$

So $A(t,\vec{\theta}) = \det\left(\langle \partial_i, \partial_j \rangle_{g(t;0,\cdots,0)}\right)_{i,j\geq 2}$

But

$$g_{ij}(t,0,\cdots,0) = \delta_{ij} - \frac{1}{3} R_{ij} t^2 + O(t^3).$$

So $A(t,\vec{\theta}) = 1 - \frac{1}{3} s(p) t^2 + O(t^3).$

$$A'(t,\vec{\theta}) = -\frac{2}{3} s(p) t + O(t^2).$$

$$\Rightarrow \lim_{t\to 0^+} \frac{A'(t,\vec{\theta})}{A(t,\vec{\theta})} = 0.$$

But $\Delta\rho(r(t)) = \dfrac{A'(t,\vec{\theta})}{A(t,\vec{\theta})} + \dfrac{n-1}{t}.$

So $\lim_{t\to 0^+}\left(u(t) - \dfrac{1}{t}\right) = 0$

On the other hand $\tilde{u}(t) = \dfrac{cs_K(t)}{sn_K(t)}.$

So $\tilde{u}(t) - \dfrac{1}{t} = \dfrac{cs_K(t)}{sn_K(t)} - \dfrac{1}{t} \longrightarrow 0.$

Rk: Laplacian comparison actually holds globally in the sense of distributions. One can prove that $\exists$ a signed measure $d\mu_{\Delta\rho}$ s.t $\forall$ $\varphi \in C^\infty(M)$,

$$\int_M \varphi \, d\mu_{\Delta\rho} = \int_M \rho \cdot \Delta\varphi$$

Then in fact, one can prove that if $Ric \geq (n-1)K$, then

$$d\mu_{\Delta\rho} \leq (n-1) \frac{sn_K'(\rho(x))}{sn_K(\rho(x))} \cdot dV_g .$$

Cor 26.4 (Myers') Let $(M, g)$ be complete w/ $Ric \geq (n-1)K$. Then

$$diam(M, g) \leq \frac{\pi}{\sqrt{K}}$$

Pf: Let $p \in M$, $q \notin Cut(M)$, $\gamma(t)$ be the ! minimal normal geod. connecting $p$ & $q$. Let $\rho(x) = d(p, x)$. Sps $\rho(q) > \pi/\sqrt{K} = t_0$. Let $\tilde{p} \in \tilde{M}_K$ & $\tilde{\gamma}$ be a normal min. geodesic. Then by Prop 26.3 $\forall$ $t < t_0$.

$$\begin{aligned} \Delta\rho(\gamma(t)) &\leq \tilde{\Delta}\tilde{\rho}(\tilde{\gamma}(t)) \\ &= (n-1) \frac{sn_K'(t)}{sn_K(t)} \end{aligned}$$

Since $\gamma(t_0) \notin Cut(p)$, $\Delta\rho^{(\gamma(t))}$ is smooth at $t = t_0$.

In particular $\lim_{t \to t_0} \Delta \rho(r(t))$ exists & is finite On the other hand, if $K > 0$, 

then $\lim_{t \to t_0^-} \tilde{\Delta} \tilde{\rho}(\tilde{r}(t)) = (n-1) \lim_{t \to t_0^-} \frac{\cos(\sqrt{K} t)}{\sin(\sqrt{K} t)} = -\infty$.

Contradiction. So $\rho(q) \leq \pi / \sqrt{K}$.