Lecture - 27

• **Rough and Hodge Laplacians** Let (M^n, g) be a Riemannian manifold w/ L-C conn. ∇

Def^n. The adjoint to $d : A^p \longrightarrow A^{p+1}$ is an operator denoted by $d^* : A^{p+1} \longrightarrow A^p$ defined by

$$(d^*\omega)_{j_1 \cdots j_p} := -\nabla^j \omega_{j j_1 \cdots j_p} = g^{ij} \nabla_i \omega_{j \cdots j_p}$$

Rk: 1) If $\omega \in A^1$, then

$$d^*\omega = -\operatorname{div}(\omega^\#).$$

To see this, note

$$-\operatorname{div}(\omega^\#) = -\nabla_i (\omega^\#)^i = -\nabla_i g^{ij} \cdot \omega_j$$

$$= -\nabla^j \omega_j = d^*\omega$$

2) $d^*\omega$ is again a form. To see this note

$$d^*\omega(X_1, \ldots, X_p) = -\operatorname{tr}(X \longrightarrow \nabla_X \omega(\cdot, X_1, \ldots, X_p))$$

So clearly $d^*\omega(X_1, \ldots X_p) = 0$ whenever $X_i = X_j$ for some $i \neq j$

Lemma 27.1 $\forall\ \alpha \in A^p$, $\beta \in A^{p+1}$, if M is compact,

$$\int \langle d\alpha, \beta \rangle = \int \langle \alpha, d^*\beta \rangle.$$

<u>"Idea of the proof"</u>. We can define a 1-form

$$\langle \alpha, \beta \rangle (X) := \langle \alpha, i_X \beta \rangle .$$

Then one can show that

$$\operatorname{div} \langle \alpha, \beta \rangle^{\#} = \langle d\alpha, \beta \rangle - \langle \alpha, d^* \beta \rangle .$$

The Lemma then follows from the divergence theorem.

<u>Def^n</u> 1) The <u>Hodge Laplacian</u> $\Delta_d : A^p \longrightarrow A^p$ is the operator defined by

$$\Delta_d := d d^* + d^* d$$

2) The rough or Bochner Laplacian is defined by

$$\begin{aligned} \Delta_B \omega &= -\operatorname{Tr} \nabla^2 \omega \\ &= -g^{ij} \nabla_i \nabla_j \omega \end{aligned}$$

<u>Rk</u>: 1) If $f \in C^\infty(M)$, clearly $\Delta_B f = -\Delta f$ where Δ is the usual Laplace Beltrami. On the other hand since $d^* f = 0$

$$\begin{aligned} \Delta_d f = d^* d f &= -\operatorname{div} (df)^{\#} \\ &= -\operatorname{div} (\nabla f) \\ &= -\Delta f \end{aligned}$$

So $\Delta_B f = \Delta_d f = -\Delta f$.

2). Δ_d is a self adjoint operator since

$$\int \langle \Delta_d \alpha, \beta \rangle = \int \langle d d^* \alpha, \beta \rangle + \int \langle d^* d\alpha, \beta \rangle$$

$$= \int \langle d^* \alpha, d^* \beta \rangle + \int \langle d\alpha, d\beta \rangle$$

$$= \int \langle \alpha, \Delta_d \beta \rangle.$$

Moreover it is non-negative since by line (2),

$$\int \langle \Delta_d \alpha, \alpha \rangle = \int \| d^* \alpha \|^2 + \int \| d\alpha \|^2 \geqslant 0$$

3) Δ_B is <u>not</u> self adjoint. In fact one can show

$$\int \langle \Delta_B \alpha, \beta \rangle = \int \langle \nabla \alpha, \nabla \beta \rangle + \int \langle \alpha, \Delta_B \beta \rangle$$

4) If we interpret $\nabla : A^p \longrightarrow \Gamma(T^*M \otimes \Lambda^p M)$ one can define an "adjoint" $\nabla^* : \Gamma(T^*M \otimes \Lambda^p M) \longrightarrow A^p$. Then in fact

$$\Delta_B \alpha = \nabla^* \nabla \alpha.$$

• <u>OVERVIEW OF HODGE THEORY</u>

<u>Def^n</u> A form $\omega \in A^p$ is said to be <u>harmonic</u> if $\Delta_d \omega = 0$.

From now on sps <u>M is compact</u>

Rk $\Delta_d \omega = 0 \iff d\omega = 0,\ d^*\omega = 0.$

We denote

$$\mathcal{H}^p = \{\omega \in \mathcal{A}^p \mid \Delta_d \omega = 0\} \leftarrow \text{vector sp.}$$

Thm 27.2 (Hodge) 1) $\dim \mathcal{H}^p(M) < \infty \ \forall p$.

2) If $\mathcal{H} : \mathcal{A}^p \longrightarrow \mathcal{H}^p$ is the orthogonal proj.
Then $\exists$ a "Greens operator" $G : \mathcal{A}^p \longrightarrow \mathcal{A}^p$
s.t. $dG = Gd$ & $d^*G = Gd^*$ & $\forall\ \omega \in \mathcal{A}^p$

$$\boxed{\omega = \mathcal{H}(\omega) + \Delta G\omega}$$

Recall:

Defn: The p^{th} De Rham cohomology group is defined to be

$$H^p_{DR}(M) := \frac{\{\omega \in \mathcal{A}^p \mid d\omega = 0\}}{\{\omega \in \mathcal{A}^p \mid \omega = d\eta\}} = \frac{\text{"closed forms"}}{\text{"exact forms"}}$$

Cor 27.3 $H^p_{DR}(M) \cong \mathcal{H}^p(M)$. In particular $H^p_{DR}(M)$ is f.d.

Defn. The p^{th} Betti # is defined by

$$\beta_p = \dim_{\mathbb{R}} H^p_{DR}(M)$$

Pf: Consider the map $\phi : \mathcal{H}^p \longrightarrow H^p_{DR}(M)$.

$$\phi(\omega) = [\omega].$$

Claim 1 ϕ is well defined

Pf: $\bar{\Delta}\omega = 0 \Rightarrow d\omega = 0$. So ω is closed & ϕ is defined.

Claim 2 ϕ is injective.

Pf: Sps $\phi(\omega) = 0 \in H^p_{DR}$ i.e $\omega = d\eta$ for some η. Then $\Delta_d \omega = 0 \Rightarrow d^*\omega = 0$

$\Rightarrow d^* d\eta = 0$. But then

$$0 = \int \langle d^* d\eta, \eta \rangle = \int |d\eta|^2$$

So $d\eta = 0$ or $\omega = 0$ (Idea: $\mathrm{Im}(d) \perp \ker(d^*)$)

Claim 3 ϕ is surjective

Pf. Let $[\omega] \in H^p_{DR}$ where $\omega \in A^p$ s.t $d\omega = 0$.

Then Hodge decomposition $\Rightarrow$.

$$\begin{aligned} \omega &= \mathcal{H}\omega + \Delta G\omega \\ &= \mathcal{H}\omega + dd^*G\omega + d^*dG\omega \\ &= \mathcal{H}\omega + dd^*G\omega \quad (\text{since } dG\omega = Gd\omega = 0) \end{aligned}$$

So $[\omega] = [\mathcal{H}\omega]$ & hence $\phi(\mathcal{H}\omega) = [\omega]$.

<u>Example</u>: Let $M^n = \mathbb{R}^n/\Lambda$. For simplicity we take $\Lambda = \mathrm{sp}_{\mathbb{Z}^n}(e_1, \dots, e_n)$. Endow M w/ the flat Euclidean metric. For any $\omega \in A^p(M)$, if $\omega = \sum_{j_1 < \dots < j_p} \omega_{j_1 \dots j_p} dx^{j_1} \wedge \dots \wedge dx^{j_p}$. then

$$\Delta_d \omega = \sum_{j_1 < \dots < j_p} (\Delta \omega_{j_1 \dots j_p}) dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

So $\Delta_d \omega = 0 \iff \Delta \omega_{j_1 \dots j_p} = 0$ on the fundamental domain. Periodicity $\Rightarrow$ $\omega_{j_1 \dots j_p}$ extends to a bounded harmonic function on $\mathbb{R}^n$. Liouville $\Rightarrow$ $\omega_{j_1 \dots j_p}$ is const. So

$$H^p_{DR}(M) \cong \mathcal{H}^p(M) \cong \Lambda^p \mathbb{R}^n$$

So $\beta_p(M) = \binom{n}{p}$.

· Bochner – Weitzenboch Formula

<u>Th^m 27.4</u>. For any $\omega \in A^p$,

$$\Delta_d \omega = \Delta_B \omega + \{R, \omega\}$$

where $\{R, \omega\}$ is a linear zero-order term involving curvature tensor. More explicitly if $\{e_i\}$ is an o.n.f for T_pM & φ^i the dual frame, then

$$\{R, \omega\} = \varphi^i \wedge i_{e_j} R(e_j, e_i)(\omega).$$

In particular if $\omega \in A^1(M)$, then

$$\Delta_d \omega = \Delta_B \omega + Ric(\omega^\#, \quad).$$

To prove this we first need the foll. formulae.

<u>Lemma 27.5</u> If e_i is an o.n.f for T_pM & φ^i dual frame, then

(1) $d = \varphi^i \wedge \nabla_{e_i}$

(2) $d^* = - i_{e_j} \nabla_{e_j}$.

(3) $\Delta_B \omega = \sum \left(\nabla_{\nabla_{e_i} e_i} \omega - \nabla_{e_i} \nabla_{e_i} \omega \right)$

<u>Pf</u>. In each case one can check that R.H.S. is independent of the o.n.b. So we pick normal coordinates at p & let $e_i = \partial_i$, $\varphi^i = dx^i$

(1) Let $\eta = f\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$. Then at p

$$dx^i \wedge \nabla_{\partial_i}(\eta) = dx^i \wedge \frac{\partial f}{\partial x^i} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$= d\eta$$

since $\nabla_{\partial_i}(dx^j)(p) = \Gamma^j_{ik}(p)\, dx^k = 0$.

(2) Again if $\eta = f\, dx^{i_0} \wedge \dots \wedge dx^{i_p}$. Then at p.

$$\nabla \eta = \frac{\partial f}{\partial x^j}\, dx^j \otimes dx^{i_0} \wedge \dots \wedge dx^{i_p}$$

So $\displaystyle d^*\eta = \sum_{k=0}^{p} (d^*\eta)_{i_0 \dots \hat{i}_k \dots i_p}\, dx^{i_0} \wedge \dots \wedge \widehat{dx^{i_k}} \wedge \dots \wedge dx^{i_p}$

$$= \sum_{k=0}^{p} (-1)^k (\nabla_{i_k} \eta)_{i_0 \cdots \hat{i_k} \cdots i_p} dx^{i_0} \wedge \cdots \wedge dx^{i_p}$$

$$= \sum_{k=0}^{p} (-1)^k (\partial_{i_k} f)\, dx^{i_0} \wedge \cdots \widehat{dx^{i_k}} \wedge \cdots \wedge dx^{i_p}$$

But $$i_{\partial_j} \nabla_{\partial_j} \eta = - \sum_{k=0}^{p} (-1)^{k-1} (\partial_{i_k} f)\, dx^{i_0} \wedge \cdots \wedge \widehat{dx^{i_k}} \cdots \wedge dx^{i_p}$$

$$= - d^* \eta$$

(3) $$\Delta_B \omega(e_{j_1}, \ldots, e_{j_p}) = -\sum_i \nabla^2 \omega(e_i, e_i, e_{j_1}, \ldots, e_{j_p})$$

$$= -\sum_i (\nabla_{e_i} \nabla \omega)(e_i, e_{j_1}, \ldots, e_{j_p})$$

$$= -\sum_i e_i(\nabla \omega(e_i, e_{j_1}, \ldots, e_{j_p}))$$

$$+ \sum_i \nabla \omega(\nabla_{e_i} e_i, \ldots)$$

$$+ \sum_{i,k} \nabla \omega(e_i, \ldots, \nabla_{e_i} e_{j_k}, \ldots)$$

$$= -\sum_i e_i(\nabla \omega(e_i, e_{j_1}, \ldots, e_{j_p}))$$

On the other hand $\nabla_{e_i} e_i = 0$ at p, so

$$(\nabla_{\nabla_{e_i} e_i} \omega - \nabla_{e_i} \nabla_{e_i} \omega)(e_{j_1} \ldots e_{j_p})$$

$$= - \nabla_{e_i} \nabla_{e_i} \omega(e_{j_1}, \ldots, e_{j_p})$$

$$= - e_i(\nabla_{e_i} \omega(e_{j_1}, \ldots e_{j_p}))$$

$$+ \nabla_{e_i} \omega(\ldots \nabla_{e_i} e_{j_k}, \ldots e_{j_p})$$

$$= - e_i(\nabla\omega(e_i, e_{j_1}, \dots e_{j_p}))$$

Pf of Th^m:

$$\Delta_d\omega = dd^*\omega + d^*d\omega$$

$$= \varphi^i \wedge \nabla_{e_i}(d^*\omega) - i_{e_j}\nabla_{e_j}(d\omega)$$

$$= -\varphi^i \wedge \nabla_{e_i}(i_{e_j}\nabla_{e_j}\omega) - i_{e_j}\nabla_{e_j}(\varphi^i \wedge \nabla_{e_i}\omega)$$

$$= -\varphi^i \wedge i_{e_j}\nabla_{e_i}\nabla_{e_j}\omega - i_{e_j}(\varphi^i \wedge \nabla_{e_j}\nabla_{e_i}\omega)$$

$$= -\varphi^i \wedge i_{e_j}\nabla_{e_i}\nabla_{e_j}\omega - \nabla_{e_i}\nabla_{e_i}\omega + \varphi^i \wedge i_{e_j}\nabla_{e_j}\nabla_{e_i}\omega$$

$$= \Delta_B\omega + \varphi^i \wedge i_{e_j}[\nabla_{e_j}, \nabla_{e_i}]\omega$$

$$= \Delta_B\omega + \varphi^i \wedge i_{e_j}R(e_j, e_i)\omega.$$

Cor 27.6 1) For $\omega \in A^p$

$$\frac{1}{2}\Delta_d|\omega|^2 = \langle\Delta_d\omega, \omega\rangle - |\nabla\omega|^2 - \langle\{R, \omega\}, \omega\rangle$$

2) When $p = 1$,

$$\frac{1}{2}\Delta_d|\omega|^2 = \langle\Delta_d\omega, \omega\rangle - |\nabla\omega|^2 - Ric(\omega^\#, \omega^\#)$$

Pf: 1) $\langle \Delta_d \omega, \omega \rangle - \langle \{R, \omega\}, \omega \rangle = \langle \Delta_B \omega, \omega \rangle$

$$= - \sum \langle \nabla_{e_i} \nabla_{e_i} \omega, \omega \rangle$$

$$= - \sum \nabla_{e_i} \langle \nabla_{e_i} \omega, \omega \rangle + |\nabla \omega|^2$$

$$= -\frac{1}{2} \sum \nabla_{e_i} \nabla_{e_i} |\omega|^2 + |\nabla \omega|^2$$

$$= \frac{1}{2} \Delta_d |\omega|^2 + |\nabla \omega|^2$$

2) When $p = 1$. One can check

$(R(X, Y)\omega)(Z) = -\omega(R(X, Y)Z)$

Then for any l.

$$\langle \{R, \varphi^l\}, \varphi^l \rangle = \langle \varphi^i \wedge i_{e_j} R(e_j, e_i) \varphi^l, \varphi^l \rangle$$

$$= i_{e_j} R(e_j, e_l) \varphi^l$$

$$= - \varphi^l (R(e_j, e_l) e_j)$$

$$= - \langle R(e_j, e_l) e_j, e_l \rangle$$

$$= -\sum_j Rm(e_j, e_l, e_j, e_l)$$

$$= Ric(e_l, e_l)$$

$$= Ric((\varphi^l)^{\#}, (\varphi^l)^{\#})$$

Cor 27.7: 1) If $\mathrm{Ric} \geq 0$, then $\beta_1 \leq \dim(M)$.

2) If $\exists\, p \in M$ s.t $\mathrm{Ric}(p) > 0$ then $\beta_1 = 0$.

Pf: 1) Integrating, we see $\forall\, \omega \in A^1$

$$\int \langle \Delta_d \omega, \omega \rangle = \int |\nabla \omega|^2 + \int \mathrm{Ric}(\omega^\#, \omega^\#).$$

So if $\omega \in \mathcal{H}^1$ & $\mathrm{Ric}\, \omega \geq 0$, then

$$\int |\nabla \omega|^2 \leq 0$$

So $\nabla \omega = 0$. Since $\|$- forms are determined by value at one-point, $\beta_1 \leq \dim T_p^* M = n$.

2) If $\mathrm{Ric}\ (p) > 0$ for some p, & $\omega \in \mathcal{H}^1$ then if $\omega \neq 0$ then

$$0 = \int |\nabla \omega|^2 + \int \mathrm{Ric}(\omega^\#, \omega^\#) > 0.$$

Contradiction. So $\mathcal{H}^1$ has only trivial form $\Rightarrow \beta_1 = 0$.

More generally recall that the curv. op. $Q : \Lambda^2 TM \to \Lambda^2 TM$ is defined by

$$\langle Q(u \wedge v), w \wedge z \rangle = \mathrm{Rm}(u, v, z, w).$$

We say $Q \geq 0$ (resp > 0) if $\forall\, s \in \Lambda^2 TM$,

$$\langle Q(s), s \rangle \geq 0 \ (\text{resp} > 0).$$

Rk $Q \geq 0$ (resp > 0) $\Longrightarrow$ $Sec \geq 0$ (resp > 0)
Converse need not be true

Cor 27.8 : 1) If $Q \geq 0$, then any harmonic form is $\parallel$. In particular, $\beta_p \leq \binom{n}{p}$.

2). If $Q \geq 0$ & $Q(p) > 0$ for some p, then $\beta_p = 0 \ \forall \ 0 < p < n$.

Pf 1).