LECTURE - 3

- RECALL. Let $E \xrightarrow{\pi} M$ be a vector bundle & $\nabla : \mathcal{X}'(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ be a connection.

Locally, if $\{e_\alpha\}$ is a basis for $E|_U$ & $\{x^i\}$ coordinates on $U \subset_{open} M$ & if $s = s^\alpha e_\alpha$, $X = X^i \partial_i$, then

$$\nabla_X s = \left(\frac{\partial s^\alpha}{\partial x^i} X^i + A_{i\beta}^\alpha \, s^\beta X^i \right) e_\alpha$$

Rk 1) Consider the bundle

$$T^*M \otimes E = \bigsqcup_p T_p^*M \otimes E_p .$$

locally generated by $\{dx^i \otimes e_\alpha\}_{i=1,\dots,n}^{\alpha=1,\dots,r}$

For any $\psi \in \Gamma(T^*M \otimes E)$, we have

$$\psi = \psi_i^\alpha \, dx^i \otimes e_\alpha$$

There is also a natural $C^\infty(M)$ bilinear <u>pairing</u>

$$\langle \, , \rangle : \Gamma(T^*M \otimes E) \times \mathcal{X}'(M) \longrightarrow \Gamma(E)$$

$$\langle \psi_i^\alpha \, dx^i \otimes e_\alpha , X^j \partial_j \rangle = \psi_i^\alpha X^i e_\alpha$$

Given ∇, $\exists !$ map $d_\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ s.t.

(1) $d_\nabla (f \cdot s) = df \otimes s + f \cdot d_\nabla s$.

(2) $\langle d_\nabla s, X \rangle = \nabla_X s$

Locally if $s = s^\alpha e_\alpha$ & $A_{i\beta}^\alpha$ are Christoffels symbols

$$d_\nabla s = \left(ds^\alpha + A_{i\beta}^\alpha dx^i \right) e_\alpha$$

We call $A_\beta^\alpha = A_{i\beta}^\alpha \, dx^i$ the <u>connection 1-form</u> of ∇ & is in fact a matrix of 1-forms $A = (A_\beta^\alpha)$.

Conversely, given a d_∇ satisfying (1), then (2) defines a connection on E.

2) (Connection NOT a tensor). A connection ∇ on E induces a map

$$A_\nabla : \mathfrak{X}(M) \times \Gamma(E) \times \Gamma(E^*) \longrightarrow C^\infty(M).$$

$$A_\nabla(X, s, \psi) = \langle \psi, \nabla_X s \rangle$$

Here $\langle \psi ; \nabla_X s \rangle$ is the natural pairing

$$\langle , \rangle : \Gamma(E) \times \Gamma(E^*) \longrightarrow C^\infty(M)$$

$$\langle \psi, t \rangle(p) = \psi_p(t(p))$$

A_∇ is NOT linear in 2^{nd} entry, so $A_\nabla \notin \Gamma(T^*M \otimes E^* \otimes E)$.

On the other hand if $\tilde{\nabla}$ is another connection

Then $A_\nabla - A_{\tilde{\nabla}} \in \Gamma(T^*M \otimes E^* \otimes E)$

$$= \Gamma(T^*M \otimes \mathrm{End}(E))$$

• <u>PARALLEL TRANSPORT</u>. Let ∇ be a connection on $E \xrightarrow{\pi} M$.

<u>Def^n</u> Let $\gamma : I \longrightarrow M$ be a C^1-regular curve. (i.e $\dot\gamma$ is cont. & $\dot\gamma \neq 0$), where $I \subseteq_{op} \mathbb{R}$ is an interval. A smooth section $s(t)$ of E along γ is a C^∞ map $s : I \longrightarrow E$ s.t $\pi \circ s = \gamma$

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ I \arrow[ur, "s"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

We denote the space of section of E along γ by $\Gamma(\gamma, E)$.

Prop 3.1 For any connection ∇ on $E \to M$ & any C^1-reg curve $\gamma : I \to M$, $\exists\,!$ operator

$$D_t : \Gamma(\gamma, E) \longrightarrow \Gamma(\gamma, E)$$

s.t.

(a) $\forall\ a_1, a_2 \in \mathbb{R}$, $s_1, s_2 \in \Gamma$

$$D_t(a_1 s_1 + a_2 s_2) = a_1 D_t s_1 + a_2 D_t s_2$$

(b) $\forall\ f \in C^\infty(I)$

$$D_t(f \cdot s) = f'(t) \cdot s + f D_t s$$

(c) If $\tilde{s}$ is any extension of s to a nbd U of $\gamma(I)$, then

$$D_t s(t) = \nabla_{\dot\gamma(t)} \tilde{s}$$

Pf: We define $D_t s(t) := \nabla_{\dot\gamma(t)} \tilde{s}$. By Lemma 2.1 this is independent of the extension.

Then (a) & (b) follow trivially from C2 & C3.

Rk: Locally if $\{e_\alpha\}$ is a basis for E & $\{x^i\}$ coordinates on $U \subset M$. If $\gamma(t_0 - \varepsilon, t_0 + \varepsilon) \subset U$. Then writing $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$; for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have

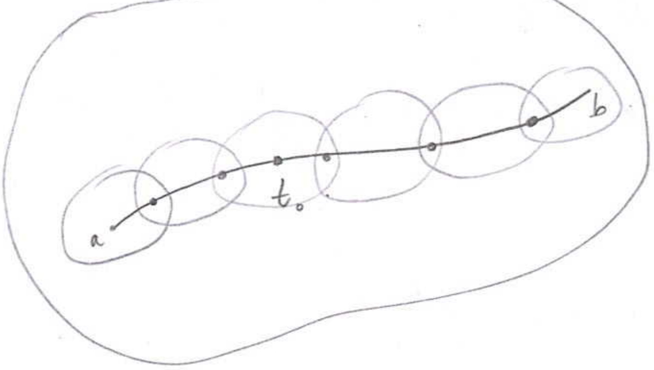
$$D_t s(t_0) = \left(\frac{ds^\alpha}{dt}(t_0) + A^\alpha_{i\beta}(r(t_0)) \cdot \dot{r}^i(t_0)\, s^\beta(t_0) \right) e_\alpha \quad (*)$$

<u>Def^n</u>. We say $s \in \Gamma(r, E)$ is <u>parallel</u> along r if $D_t s \equiv 0$.

<u>Theorem 3.2</u> Let $r : I \subset M$ be a C^1-reg curve & $t_0 \in I$. For any $v \in E_{r(t_0)}$, $\exists !$ section $s \in \Gamma(r, E)$ s.t. $s(t_0) = v$ &

$$D_t s \equiv 0$$

<u>Pf</u>:



Let U be a trivializing nbd of E, and let $\{e_\alpha\}$ be the local basis & $\{x^i\}$ coordinates. Sps $r[t_0-\varepsilon, t_0+\varepsilon] \subset U$. By a standard result for system of linear O.diff equation $\exists !$ solution $\{s^\alpha(t)\}_{\alpha=1}^{k}$ s.t.

$$\begin{cases} \dot{s}^\alpha(t) = A^\alpha_{i\beta}(r(t)) \cdot \dot{r}^i(t)\, s^\beta(t) \\ \\ s^\alpha(t_0) = v^\alpha, \quad v = v^\alpha e_\alpha(r(t_0)) \end{cases}$$

$\forall \; t \in (t_0 - \varepsilon, t_0 + \varepsilon)$

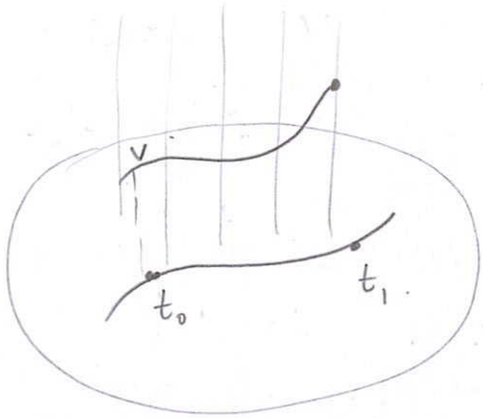
More generally if $\tilde{I} = [a,b] \subset\subset I$, $t_0 \in I$, then taking a partition $a < a_1 < \dots < a_m = b$ s.t

(1) $a_{k-1} < t_0 < a_k$ for some k

(b) $\gamma[a_{j-1}, a_j] \subset U_j \subset_{open} M$, U_j coordinate nbd & $E|_{U_j}$ is trivial.

By above $\exists$ a $\parallel$-section s on $U_k \cap \gamma(I)$. Then $\exists!$ a $\parallel$-section on U_{k-1} starting w/ initial value $s(a_{k-1})$. Similarly on U_{k+1}, an so on. By uniqueness, s is well defined on $U_j \cap U_{j'} \cap \gamma(I)$. So we get a unique $\parallel$ section on $\tilde{I}$ & hence on I (since this holds for any $\tilde{I} \subset\subset I$).

<u>Def^n</u>: Let $\gamma: I \longrightarrow M$ be a C^1-curve. We define



the parallel transport map $P_{t_0,t,\gamma} : E_{\gamma(t_0)} \longrightarrow E_{\gamma(t)}$ by the property that (a) $P_{t_0,t_0,\gamma}(v) = v$

(b) If $s(t) = P_{t_0,t,\gamma}$, then $D_t s \equiv 0$.

Lemma 3.3. $P_{t_0,t,\gamma}$ is a linear isomorphism. Moreover

$$P_{\gamma;t_0,t_1} \circ P_{\gamma;t_1,t_2} = P_{t_0,t_2,\gamma} \;\&\; P_{\gamma;t_0,t_1}^{-1} = P_{\gamma;t_1,t_0}$$

In particular if γ is a closed curve w/ $\gamma(t_0) = \gamma(t_1) = p$ Then $P(\gamma) := P_{\gamma;t_0,t_1}$ is an automorphism of E_p.

Def^n: (Holonomy) 1) If γ is a loop at p, we define the holonomy of ∇ around γ by

$$Hol_p(\gamma;\nabla) := P(\gamma) \in Aut(E_p)$$

2) The <u>holonomy group</u> is defined as

$$Hol_p(\nabla) := \{ Hol(\gamma;\nabla) \mid \gamma \text{ is a closed loop at } p \}$$

$$P(\gamma)\cdot P(\gamma') = P(\gamma\cdot\gamma')$$

3) The <u>reduced</u> holonomy group is defined as

$$Hol_p^0(\nabla) = \{ Hol(\gamma;\nabla) \mid \gamma \sim 0 \}$$

<u>Rk</u> 1) One should think of $\tilde{\gamma}_v(t) := P_{t_0,t,\gamma}(v)$ as a horizontal lift of the curve $\gamma(t)$ in the sense that $d\pi(\tilde{\gamma}_v'(t)) = \gamma'(t)$. Here $d\pi : TE \longrightarrow TM$, $\tilde{\gamma}_v'(t) \in T_{\tilde{\gamma}_v(t)}E$.

2) If M is connected, then $Hol_p(\nabla) \cong Hol_q(\nabla)$ $\forall\, p, q \in M$. The isomorphism is by conjugating by $P_{\gamma;t_0,t_1}$ where $\gamma(t_0) = p$, $\gamma(t_1) = q$.

3) $Hol^\circ(\nabla)$ = identity component of $Hol(\nabla)$.

$\lhd \; Hol(\nabla)$.

$\exists$ surjective homomorphism.

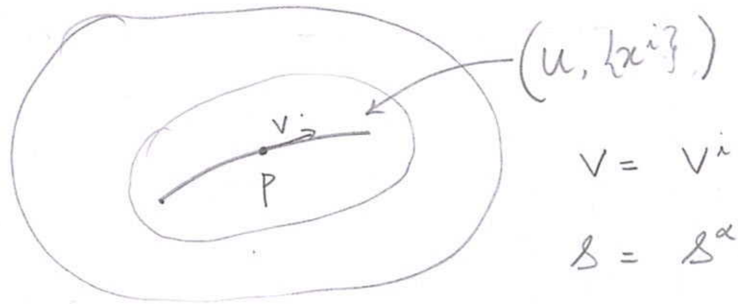
$$\rho : \pi_1(M) \longrightarrow Hol(\nabla)/Hol^\circ(\nabla).$$

• Prop. 3.4 Let ∇ be a connection on $E \xrightarrow{\pi} M$. For $p \in M$, let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ be a C^1-reg. curve s.t. $\gamma(0) = p$, $\dot\gamma(0) = V \in T_pM$. Then for any section $s \in \Gamma(E)$,

$$\nabla_V s = \lim_{h \to 0} \frac{P_{\gamma; h, 0}\, s(\gamma(h)) - s(p)}{h}$$

$$= \frac{d}{dt}\Big|_{t=0} P_{\gamma; t, 0}\, s(\gamma(t)).$$

Pf:



$V = V^i \partial_i$

$s = s^\alpha e_\alpha$.

Let $\{e_\alpha\}$ be a basis for E_p, and for $t \in (-\varepsilon, \varepsilon)$, let

$$e_\alpha(t) := P_{\gamma, 0, t}(e_\alpha) \in E_{\gamma(t)}.$$

Since $P_{\gamma, 0, t}$ is an iso $\Longrightarrow$ $\{e_\alpha(t)\}$ is a basis for $E_{\gamma(t)}$. Then if $s(\gamma(t)) = \sum s^\alpha(t)\, \underline{e_\alpha(t)}$, we have

$$D_t\, s(\gamma(t)) = \sum s_\alpha'(t) \cdot e_\alpha(t) + s^\alpha(t)\, D_t\, e_\alpha(t).$$

$$= \frac{d}{dt} s^\alpha(r(t)) \cdot e_\alpha(t).$$

In particular $\nabla_v s(p) = \left.\frac{d}{dt}\right|_{t=0} s^\alpha(r(t)) \cdot e_\alpha.$

On the other hand.

$$P_{r,t,0}\, s(r(t)) = P_{r,0,t}^{-1}\left(s^\alpha(r(t)) \cdot e_\alpha(t)\right).$$

$$= s^\alpha(r(t)) \cdot e_\alpha.$$

So $$\left.\frac{d}{dt}\right|_{t=0} P_{r,t,0}\, s(r(t)) = \left.\frac{d}{dt}\right|_{t=0} s^\alpha(r(t)) \cdot e_\alpha = \nabla_v s(p).$$

Done!