LECTURE - 4

• RIEMANNIAN METRICS

Def^n: A Riemannian metric g on M is a smooth positive definite section of S^2T^*M, the bundle of symmetric bi-linear forms. Equivalently

$$g : \mathcal{T}^1(M) \times \mathcal{T}^1(M) \longrightarrow C^\infty(M)$$

s.t (1) g is $C^\infty(M)$ - bi-linear

(2) $g(X, Y) = g(Y, X) \quad \forall \; X, Y \in \mathcal{T}^1(M)$

(3) $g_p(X_p, X_p) > 0 \quad \forall \; X_p \in T_pM \setminus \{0\}$

We say (M, g) is a Riemannian mfld.

Rk 1) If $X_p = 0$, then $g_p(X_p, Y_p) = 0 \;\forall\; Y \in \mathcal{T}^1(M)$.

since if $f \in C^\infty(M)$ s.t $f(p) = 0$, then $f(p)X_p = X_p$. so

$$g_p(X_p, Y_p) = g_p(f(p)X_p, Y_p) = f(p)\, g_p(X_p, Y_p) = 0 .$$

In particular, if $X_p = \widetilde{X}_p$ for some $X, \widetilde{X} \in \mathcal{T}^1(M)$

Then $g(X, Y)(p) = g(\widetilde{X}, Y)(p) \quad \forall \; Y \in \mathcal{T}^1(M)$.

So g defines a pointwise inner product.

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(u, v) \longmapsto g(\tilde{u}, \tilde{v})$$

where $\tilde{u}, \tilde{v}$ are any extensions to sections in $\mathcal{T}^1(M)$.

2) Given $\xi_p \in T_pM$, we define the norm by.

$$|\xi_p|^2 = g_p(\xi_p, \xi_p)$$

3) Locally, if $U \subset M$ has coordinates, then if we set

$$g_{ij}(x) = g\left(\frac{\partial}{\partial x^i}\Big|_x , \frac{\partial}{\partial x^j}\Big|_x\right)$$

then $g_{ij} : U \longrightarrow \mathbb{R}$, and for any $X = X^i \partial_i$, $Y = Y^j \partial_j$,

$$g(X, Y)\Big|_U = g_{ij} X^i X^j$$

<u>Note</u>: $\{g_{ij}(x)\}$ is a <u>symmetric</u>, <u>+ve definite</u> matrix $\forall\ x \in U$.

We also write

$$\boxed{g = g_{ij}\, dx^i \otimes dx^j}$$

In particular if $\xi = \xi^i \partial_i$

$$|\xi|^2 = g_{ij}\, \xi^i \xi^j > 0$$

unless $\xi^i = 0\ \forall\ i$.

<u>Example</u>: 1) On $\mathbb{R}^n$, we have the standard <u>Euclidean metric</u>

$$g_{Euc} = \sum_i dx^i \otimes dx^i$$

Then any $\vec{v} \in T_pM \approx \mathbb{R}^n$, $\vec{v} = (v_1, \dots, v_n)$.

$$|v|^2 = \sum (v_i)^2$$

More generally any (U, g_{Euc}) is a Riemannian mfld if $U \subset_{open} \mathbb{R}^n$

2) $\mathbb{B}^n_R = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid \sum (x^i)^2 < R\}$

$$g_{Poin} = 4R^2 \sum_i \frac{dx^i \otimes dx^i}{(R^2 - |x|^2)^2}, \quad |x|^2 = \sum (x^i)^2$$

Poincaré metric.

3) (Poincaré upper half sp).

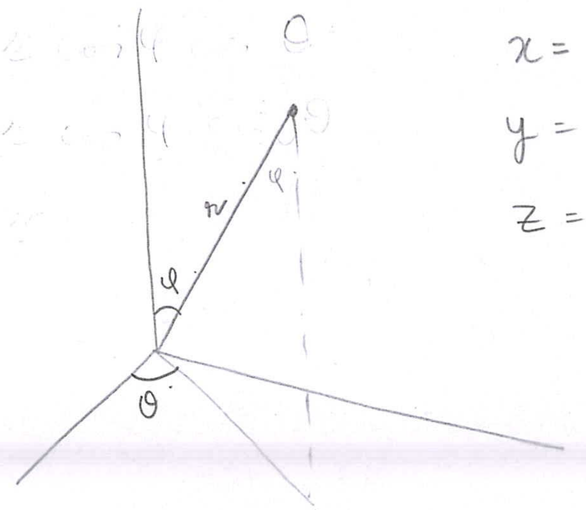
$$U^n_R = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$$

Define

$$= R^2 \sum_i \frac{dx^i \otimes dx^i}{(x^n)^2}$$

4) (Euclidean metric in polar coordinates).

On $\mathbb{R}^3$,



$x = r \sin\varphi \cos\theta$

$y = r \sin\varphi \sin\theta$

$z = r \cos\varphi$

Check: $g_{Euc} := (dx)^2 + (dy)^2 + (dz)^2$.

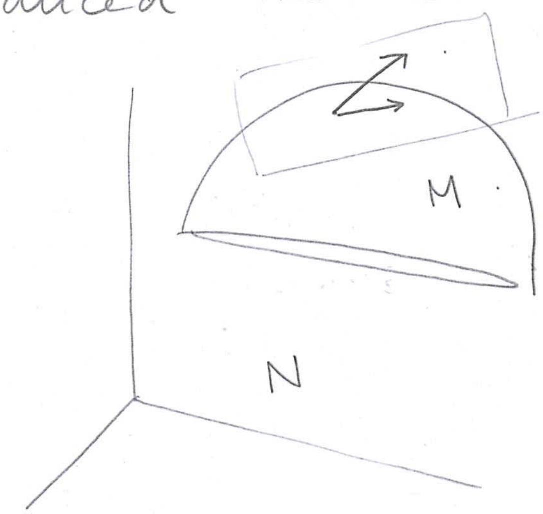
$$\boxed{g_{Euc} = (dr)^2 + r^2(\sin^2\varphi \, d\theta^2 + d\varphi^2)}$$

Lemma 4.1: 1) (Pull backs) Let (N, g) be a Riemannian manifold & $\varphi : M \longrightarrow N$ be an immersion (i.e $\ker d\varphi = \{0\}$). Then $\varphi^* g$, defined by

$$\varphi^* g(X, Y) = g(d\varphi(X), d\varphi(Y))$$

$\forall X, Y \in \mathcal{T}^1(M)$, is a metric on M.

2) In particular, if $M \subset N$ is a sub-manifold (i.e $M \subset N$ as sets & $i : M \hookrightarrow N$ is an embedding), and g is a metric on N, then $i^* g$ is a metric on M, called the restriction or induced metric.



3) (Product metric) If (M, g_M) & (N, g_N) are Riemannian manifolds, then $M \times N$ can be given the product metric $g_M \oplus g_N$ s.t for any $u_i \oplus v_i \in T_{(p,q)}(M \times N) = T_p M \oplus T_q N$

$$(g_M \oplus g_N)(u_1 \oplus v_1, u_2 \oplus v_2) = g_M(u_1, u_2) + g_N(v_1, v_2)$$

• MORE EXAMPLES.

1) Spheres. $S^n_R = \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1} \mid \sum (x^i)^2 = R^2\}$

The induced metric is denoted by g_{S^n}

FACT: $g_{Euc}\big|_{\mathbb{R}^n - \{0\}} = dr^2 + r^2 g_{S^{n-1}}$

When $n = 2$: Spherical coordinates (θ, φ) on S^2

We have seen

$$g_{\mathbb{R}^3} = dr^2 + r^2(d\varphi^2 + \sin^2\varphi \, d\theta^2).$$

So $g_{\mathbb{R}^3}\big|_{r=R}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = R^2 \sin^2\varphi$, $g_{\mathbb{R}^3}\big|_{r=R}\left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\right) = R^2$.

$$g_{\mathbb{R}^3}\big|_{r=R}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right) = 0$$

So $\boxed{g_{S^2_R} = R^2(d\varphi^2 + \sin^2\varphi \, d\theta^2)}$

2) Cylinder: $C_R = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = R\} = S^1_R \times \mathbb{R}$

Then $g_{\mathbb{R}^3}$ induces a metric g_{C_R} on C_R.

We can parametrize C_R by (θ, z),

$$x = R\cos\theta, \quad y = R\sin\theta, \quad z = z$$

$$dx\big|_{C_R} = -R\sin\theta \, d\theta, \quad dy\big|_{C_R} = R\cos\theta \, d\theta$$

$$dz\big|_{C_R} = dz$$

So $g_{C_R} = g_{\mathbb{R}^3}\big|_{C_R} = \boxed{R^2 d\theta^2 + dz^2}$.

3) General Cylinders. $C_R^{n,m} = S_R^n \times \mathbb{R}^m$ w/ the product metric

Def^n: Let (M, g) be a Riemannian mfld. Then (N, h) is a Riemannian sub-manifold if (a) N is a sub-manifold of M.

(b) $\forall\ p \in N$ h_p is the restriction of g_p to $T_pN \subset T_pM$.

• ISOMETRIES:

Def^n: Let (M, g) & (N, h) be Riemannian manifolds. A map $\varphi: M \longrightarrow N$ is called an (local) isometry (resp. homothety), and we write $(M, g) \cong (N, h)$, if

(a) φ is a (local) diffeo.

(b) $\varphi^* h = g$ (resp $\varphi^* h = \lambda g$, $\mathbb{R} \ni \lambda > 0$).

The set of isometries of (M, g) to itself forms a group called the Isometry group $\mathrm{Isom}(M, g)$.

Examples: 1) (Hyperbolic sp) $(\mathbb{B}_R^n, g_{Poin}) \cong (\mathbb{U}_R^n, g_{\mathbb{U}_R^n})$

If $n = 2$, then we can consider $\mathbb{B}^2, \mathbb{U}^2 \subset \mathbb{C}$.

Then the Cayley transform

$$K_R: \mathbb{B}_R^2 \longrightarrow \mathbb{U}_R^2$$
$$K_R(w = u + iv) = z = -iR\left(\frac{w + iR}{w - iR}\right).$$

i.e. $\kappa(u,v) = \left(\dfrac{2R^2 u}{|u|^2 + (v-R)^2} , R \dfrac{R - |u|^2 - |v|^2}{|u|^2 + (v-R)^2} \right)$.

More generally, if we let $\mathbb{B}^n_R = \{ (u^1, \dots, u^{n-1}, v) \mid |u|^2 + |v|^2 < R \}$

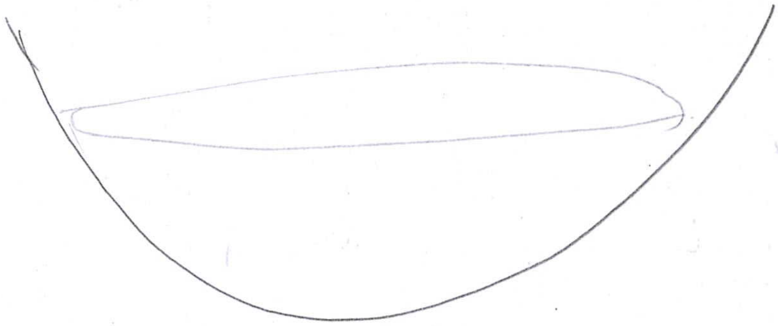
Then we define $\kappa_R : \mathbb{B}^n_R \longrightarrow \mathbb{U}^n_R$ by

$$\kappa_R(u,v) = \left(\frac{2R^2 u}{|u|^2 + (v-R)^2} , R \frac{R - |u|^2 - |v|^2}{|u|^2 + (v-R)^2} \right).$$

Note that κ_R is obtained from κ_1 by setting

$$\kappa_R(u,v) = R \kappa_1 \left(\frac{u}{R}, \frac{v}{R} \right).$$

There is a <u>third</u> <u>model</u>: hyperboloid model



$\mathbb{H}^n_R = \{ (x^0, \dots, x^n) \in \mathbb{R}^n \mid \langle x, x \rangle := -x_0^2 + x_1^2 + \dots + x_n^2 = -R^2, \ x_0 > 0 \}$.

(of sig $(n,1)$).

The quadratic form $\langle , \rangle$ induces a +ve definite Riemannian metric on $\mathbb{H}^n$ denoted by $g_{\mathbb{H}^n}$.

<u>FACT</u>: $(\mathbb{H}^n_R, g_{\mathbb{H}^n}) \cong (\mathbb{B}^n, g_{\mathbb{B}^n}) \cong (\mathbb{U}^n_R, g_{\mathbb{U}^n_R})$.

2). $\mathrm{Isom}(\mathbb{R}^n, g_{Euc}) =: E(n)$.

$$\varphi \in E(n) \iff \exists!\, A \in O(n),\ \vec{c} \in \mathbb{R}^n \text{ s.t}$$

$$\varphi(\vec{x}) = A\vec{x} + \vec{c}.$$

$$T(n) = \{ \vec{x} \longrightarrow \vec{x} + \vec{c} \} \triangleleft E(n).$$

In fact $\quad E(n) = O(n) \ltimes T(n)$.

• Riemannian Coverings & Quotients

Def^n: Let (M, g) & (N, h) be Riemannian manifolds. A map $p : N \to M$ is a Riemannian covering map if

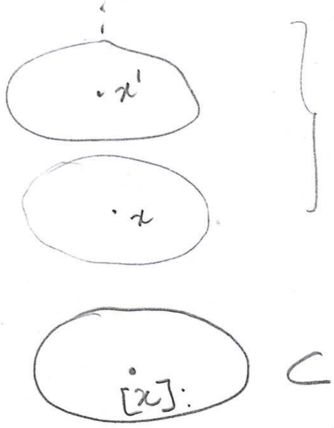
(a) p is a smooth covering map

(b) p is a local isometry.

Lemma 4.2. 1) Let $p : N \longrightarrow M$ be a smooth covering map. For any metric g on M, $\exists!\ h$ on N s.t p is a Riemannian cover.

2) Let (N, h) be a Riemannian mfld & $G \subset \mathrm{Isom}(N, h)$ act freely & properly discont. on N. Then $\exists!\ g$ on $M = N/G$ s.t $p : N \longrightarrow M$ is a Riemannian covering map.

Pf (2)



$\subset M$.

For, $[x] \in M$, $u, v \in T_{[x]} M$, we define

$$g_{[x]}(u, v) = h_x((d_x p)^{-1} u, (d_x p)^{-1} v).$$

Note p is a local diffeo from a nbd of $x \in N$ to a nbd of $[x] \in M$. So above eq is defined. If $x' \in [x]$, then $x' = \varphi(x)$ for some $\varphi \in G$. Then $(d_{x'} p)^{-1} = (d_x \varphi) \circ (d_x p)^{-1}$. Since $\varphi \in G$, $d_x \varphi$ is an isometry between $(T_x N, h_x)$ & $(T_{x'} N, h_{x'})$. So $g_{[x]}$ is well defined. It is also smooth.

Examples. 1) Flat tori: Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be any basis of $\mathbb{R}^n$, and let

$$\Lambda = \{ \sum k_i \vec{v}_i \mid k_i \in \mathbb{Z} \}.$$

Then $\Lambda \cong \mathbb{Z}^n$ & $\Lambda \curvearrowright \mathbb{R}^n$ by translations. Action is properly discont. & free. $g_{Euc} = \sum_i dx^i \otimes dx^i$ is translation inv., so $\Lambda \subset \text{Isom}(\mathbb{R}^n, g_{Euc})$. Then

$$\mathbb{T}^n_\Lambda = \mathbb{R}^n / \Lambda$$

is canonically equipped w/ a metric g_Λ.

Note $\mathbb{T}^n_\Lambda \overset{\text{differ}}{\cong} \mathbb{T}^n = (S^1)^n$ via

$$p : \mathbb{T}^n_\Lambda \longrightarrow \mathbb{T}^n$$

$$[\sum x_j \vec{v}_j] = (e^{2\pi i x_j})$$

FACT: $(\mathbb{T}^n, g_\Lambda) \cong (\mathbb{T}^n, g_{\Lambda'}) \iff \exists\ \varphi \in \text{Isom}(\mathbb{R}^n, g_{Euc})$ s.t. $\varphi(\Lambda) = \Lambda'$.