LECTURE - 5

• Recall: A Riemannian metric g on M is a smooth, +ve definite section of $S^2 T^* M$.

Locally

$$g = g_{ij}\, dx^i \otimes dx^j .$$

• Existence

Prop 5.1 On any manifold M, there exists at least one Riemannian metric.

The proof relies on the foll Lemma.

Lemma 5.2. (Partitions of unity) Given any open, locally finite cover $\{U_\alpha\}_{\alpha \in I}$ of M, $\exists$ a partition of unity $\{\rho_\alpha\}$ sub-ordinate to the cover. i.e $\forall\ \alpha \in I$, $\exists$

$\rho_\alpha : M \longrightarrow \mathbb{R}$ s.t

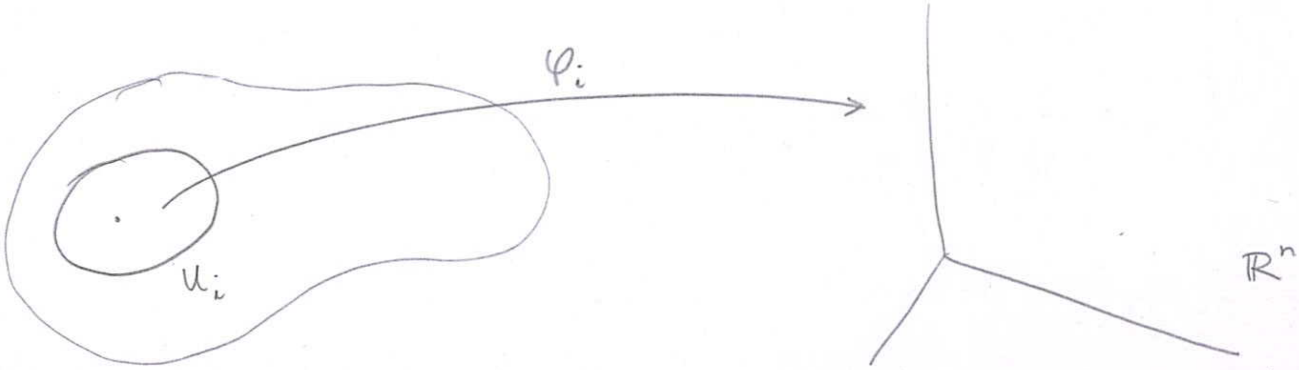
(1) $0 \leq \rho_\alpha \leq 1$

(2) $\operatorname{supp} \rho_\alpha := \overline{\{x \in M \mid \rho_\alpha(x) \neq 0\}} \subseteq U_\alpha$

(3) $\underbrace{\sum_{\alpha \in I}}_{\text{finite sum}} \rho_\alpha(x) = 1 \quad \forall\ x \in M.$

You can find the proof in "Smooth manifolds" by Lee.

Pf of Prop: Let $\{U_i\}$ be a locally finite cover of M by coordinate neighbourhoods & let $\varphi_i : U_i \longrightarrow \mathbb{R}$ be the corresponding charts.



Let $\{\rho_i\}$ be a partition of unity sub-ordinate to $\{U_i\}$. Set

$$g := \sum \rho_i(x) \cdot \varphi_i^* g_{\mathbb{R}^n}$$

Claim g is a Riemannian metric on M.

Pf: Clearly g is a smooth section of S^2T^*M.

Let $\xi \in T_pM$. Then

$$g_p(\xi, \xi) = \sum_i \rho_i(p) \cdot g_{\mathbb{R}^n}(d_p\varphi_i(\xi), d_p\varphi_i(\xi))$$

If $\xi \neq 0$, then $d_p\varphi_i(\xi) \neq 0 \ \forall i$, and since $g_{\mathbb{R}^n}$ is +ve definite, # of i s.t $\rho_i(p) \neq 0$ is finite, $\exists \ \lambda > 0$ s.t $\forall \ i$ w/ $\rho_i(p) \neq 0$ we have $g_{\mathbb{R}^n}(d_p\varphi_i(\xi), d_p\varphi_i(\xi)) > \lambda$.

So $\quad g_p(\xi, \xi) > \lambda \sum_i \rho_i = \lambda > 0$

Hence g is also positive def^n

<u>Rk</u>: This crucially relies on the fact that $g_{\mathbb{R}^n}$ is +ve def^n. For instance, on S^2 there is no Lorentzian metric (i.e a section of S^2T^*M of signature $(1,-1)$).

• <u>SOME ELEMENTARY CONSTRUCTIONS</u> (M,g) R-mfld.

1) <u>Raising & Lowering indices</u>:

$\exists$ a map, called <u>flat</u>, $\flat : TM \longrightarrow T^*M$ defined by

$$\boxed{X^\flat(Y) = g(X,Y)}$$

Locally, if $X = X^i \partial_i$, then

$$X^\flat = X^i g_{ij}\, dx^j$$

i.e if $\omega = X^\flat$, then $\omega = \omega_j dx^j$, where $\omega_j = X^i g_{ij}$. We say $X^\flat$ is obtained from X by lowering an index

<u>Prop 5.3</u> $\flat : TM \longrightarrow T^*M$ is an isomorphism.

The inverse, called the <u>sharp operator</u> $\# : T^*M \to TM$ is locally given by

$$(\omega_j dx^j)^\# = g^{ij}\omega_j \frac{\partial}{\partial x^i},$$

where $\{g^{ij}\}$ is the inverse of $\{g_{ij}\}$ & is <u>also</u> a matrix of smooth forms since $\det(g_{ij}) > 0$.

"Pf" For each $p \in M$, $\flat : T_pM \longrightarrow T_p^*M$ has trivial kernel since if $\xi \in \ker(\flat|_{T_pM})$, then $\forall\ Y \in T_pM$, $\xi^\flat(Y) = g(\xi, Y) = 0$.

g +ve def^n $\Longrightarrow$ $\xi = 0$.

Also $\dim T_pM = \dim T_p^*M$, so clearly $\flat$ is an isomorphism. So $\forall\ p \in M$ we have $\# : T_p^*M \to T_pM$ s.t $(X^\flat)^\# = X$ & $(\omega^\#)^\flat = \omega$. If $\{x^i\}$ are coordinates near p, then since if $\omega = \omega_j dx^j$, since $(\omega^\#)^\flat = \omega$, we have

$$(\omega^\#)^k g_{kj}\, dx^j = \omega_j .$$

So $$(\omega^\#)^k g_{kj} = \omega_j .$$

$$\Longrightarrow \quad (\omega^\#)^k \underbrace{g_{kj} \cdot g^{ji}}_{\delta^k_i} = \omega_j g^{ji} \Longrightarrow (\omega^\#)^i = \omega_j g^{ji}$$

Since g^{ji} is smooth, $\#$ is a bundle morphism

<u>Def^n</u>: Given $f \in C^\infty(M)$, we define the <u>gradient</u> of f, by
$$\nabla f = (df)^\# \in \mathcal{T}^1(M)$$

Locally $\boxed{\nabla f = g^{ij} \dfrac{\partial f}{\partial x^j} \dfrac{\partial}{\partial x^i}}$, since $df = \dfrac{\partial f}{\partial x^j} dx^j$

<u>Rk</u>: ① If M is connected, then $\nabla f \equiv 0 \Longleftrightarrow f = \text{const}$.

② One can use $\flat$ & $\#$ to lower & raise indices of other tensors. For instance if $T = T_i{}^l{}_k \, dx^i \otimes \partial_l \otimes dx^k$, we can lower the 2^{nd} index & define $T^\flat = T_{ijk} \, dx^i \otimes dx^j \otimes dx^k$ where $T_{ijk} = g_{lj} T_i{}^l{}_k$. More invariantly,

$$T^\flat(X, Y, Z) = T(X, Y^\flat, Z)$$

Of course if there are multiple upper & lower indices, one can raise & lower multiple indices e.g. $T_i{}^j{}_k{}^l = g^{jp} T_{ip}{}^{ql} g_{qk}$.

③ <u>Trace w.r.t g</u>. If $h \in \Gamma(S^2 T^* M)$, we can define $h^\# \in \Gamma(\mathrm{End}\, TM)$ by $(h^\#)_i{}^j = h_{ik} g^{kj}$.

We the define

$$\boxed{tr_g h := tr(h^\#) = h_{ij} \, g^{ij}}$$

2) <u>Inner product on tensors</u>

<u>Def^n</u> Let $\pi : E \to M$ be a v.b. Then a <u>fibre metric</u> on E is an <u>inner product</u> $\langle , \rangle_p$ on each E_p, s.t it varies smoothly in the sense that $\forall \; \sigma, \tau \in \Gamma(E)$, $\langle \sigma, \tau \rangle \in C^\infty(M)$.

A Riemannian metric g on M induces a fiber metric on all tensor bundles $T^r_s M$ in the foll way.

1) $\underline{r = 0, s = 1}$: If $\omega_p, \eta_p \in T_p^* M$, we define

$$\langle \omega_p, \eta_p \rangle = g(\omega_p^\#, \eta_p^\#).$$

2) In general, if $\{E_i\}$ is basis for T_pM & $\{\varphi^i\}$ a dual basis for T_p^*M, we define

$$\langle\, , \rangle : T_pM^{\otimes r} \otimes T_p^*M^{\otimes s} \longrightarrow \mathbb{R}$$

by $\langle E_{i_1} \otimes \cdots \otimes E_{i_r} \otimes \varphi^{j_1} \otimes \cdots \otimes \varphi^{j_s}, E_{k_1} \otimes \cdots \otimes \varphi^{l_s} \rangle$

$$= \langle E_{i_1}, E_{k_1} \rangle \langle E_{i_2}, E_{k_2} \rangle \cdots \langle \varphi^{j_1}, \varphi^{l_1} \rangle \cdots \langle \varphi^{j_s}, \varphi^{l_s} \rangle$$

and extending bilinearly.

<u>Claim</u> The above fiber-wise metric is smooth

<u>Pf</u>: Locally if $\{x^i\}$ is a coordinate system on $U \subset_{open} M$.

$$\langle dx^i, dx^j \rangle := g((dx^i)^\#, (dx^j)^\#)$$
$$= g(g^{ik} \partial_k, g^{jl} \partial_l)$$
$$= g^{ik} g^{jl} g_{kl}$$
$$= g^{ik} \delta^j_k$$

i.e.

$$\boxed{\langle dx^i, dx^j \rangle = g^{ij}} \in C^\infty(U).$$

More generally, if

$$\langle \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s},\ \partial_{k_1} \otimes \cdots \otimes dx^{l_s} \rangle$$

$$:= g_{i_1 k_1} g_{i_2 k_2} \cdots g_{i_r k_r} g^{j_1 l_1} \cdots g^{j_s l_s} \in C^\infty(U).$$

Rk: The above proof shows that if $T, S \in \mathcal{T}^r_s(M)$ then

$$\langle T, S \rangle = T^{i_1 \cdots i_r}_{j_1 \cdots j_s} S^{k_1 \cdots k_r}_{l_1 \cdots l_s} g_{i_1 k_1} g_{i_2 k_2} \cdots g^{j_1 k_1} \cdots g^{j_s l_s}.$$

- Volume Element & Integration

Def^n (Oriented manifold). A mfld is called orientable if $\exists$ cover $\{U_\alpha\}$ by coordinate neighbourhoods w/ coordinates $\{x^i_\alpha\}$ s.t whenever $U_\alpha \cap U_\beta \neq \emptyset$,

$$\det\left\{ \frac{\partial x^i_\alpha}{\partial x^j_\beta} \right\} > 0.$$

M w/ the additional data of a cover as above, is said to be an oriented mfld. Any other cover $\{\psi_\beta, \{y^i_\beta\}\}$ gives same orientation if $\det\{\frac{\partial x^i_\alpha}{\partial y^j_\beta}\} > 0$.

Prop 5.4 M^n is orientable $\iff \exists\ \Omega \in A^n(M^n)$ s.t $\Omega(p) \neq 0\ \forall\ p \in M$.

Outline of proof: $\Rightarrow$ If $\{U_\alpha, \{x^i_\alpha\}\}$ is a coordinate chart as above & $\{\rho_\alpha\}$ is a partition of unity sub-ordinate to $\{U_\alpha\}$, set

$$\Omega = \sum \rho_\alpha(x) \cdot dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$$

$\Longleftarrow$: Choose coordinates $\{x_\alpha^i\}$ on $U_\alpha \subset$ open M s.t

$$\Omega = f_\alpha \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

s.t $f_\alpha > 0$. Then

$$f_\beta = f_\alpha \cdot \det\left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right).$$

So $\det(\partial x_\alpha^i / \partial x_\beta^j) > 0$.

<u>Prop 5.5</u>: Let (M, g) be an <u>oriented</u>, Riemannian manifold w/ a oriented cover $\{U_\alpha, \{x_\alpha^i\}\}$. Then if $g_{ij}^{(\alpha)} = g(\partial_{x_\alpha^i}, \partial_{x_\alpha^j})$,

$$dV_g = \sqrt{\det\{g_{ij}^\alpha\}} \; dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

is a well defined, nowhere vanishing section in $A^n(M)$, and hence defines a measure on M. We call dV_g, the <u>volume form</u> corresponding to the metric g.

Here $\sqrt{\det}$ is the +ve square-root.

<u>Def$^{n.}$</u> (L^2-norm) For any $S, T \in \mathcal{T}_s^r(M)$, we define

$$\langle S, T \rangle := \int_M \langle S, T \rangle_p \, dV_g(p).$$

<u>Rk</u>: $\mathcal{T}_s^r(M)$ w/ the norm $\|T\| = \langle T, T \rangle^{1/2}$ is a normed linear space.