• LEVI - CIVITA CONNECTION :

Def^n: Let ∇ be a linear connection on (M, g).

1) We say ∇ is compatible with the metric g if $\forall\, X, Y, Z \in \mathcal{T}^1(M)$,
$$Z\, g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$
$$(\Longleftrightarrow \; d\, g(X,Y) = g(d_\nabla X, Y) + g(X, d_\nabla Y)).$$

2) We say ∇ is symmetric if $\forall\, X, Y \in \mathcal{T}^1(M)$.
$$\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \equiv 0.$$

Rk 1) ∇ being metric compatible $\Longleftrightarrow$ $\nabla g \equiv 0$.
To see this, note that for any $X, Y, Z \in \mathcal{T}^1(M)$,
$$\nabla_Z g\,(X,Y) = Z\, g(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y).$$

2) τ defines a tensor. i.e $\tau : \mathcal{T}^1(M) \times \mathcal{T}^1(M) \longrightarrow \mathcal{T}^1(M)$ is $C^\infty(M)$ - bilinear. We can interpret $\tilde{\tau} \in \mathcal{T}^1_2(M)$ by setting
$$\tilde{\tau}(X_1, X_2, \omega^1) = \omega^1(\tau(X_1, X_2))$$

Th^m 6.1/Def^n. (Fundamental Theorem of Riemannian geom). On any (M, g), $\exists$ a unique symmetric metric compatible connection ∇, called the Levi - Civita connection.

2

Pf: • Uniqueness Sps $\exists$ such a connection ∇.

Then $\nabla g = 0 \Rightarrow \forall X, Y, Z \in \mathcal{T}^1(M)$,

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad (1)$$

$$Y g(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \qquad (2)$$

$$Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \qquad (3)$$

Adding (1) + (2) − (3), & using symmetry of ∇ & g.

$$X g(Y, Z) + Y g(Z, X) - Z g(X, Y)$$
$$= 2 g(\nabla_X Y, Z) - g([X, Y], Z) + g([X, Z], Y) + g([Y, Z], X).$$

So

$$\boxed{2 g(\nabla_X Y, Z) = X g(Y, Z) + Y g(X, Z) - Z g(X, Y) - g(Y, [X, Z]) - g(X, [Y, Z]) + g(Z, [X, Y])}$$

Koszul-formula.

Since g is non-degenerate if ∇' is another such connection, then $\nabla_X Y = \nabla'_X Y \ \forall X, Y$.

• Existence: Given $X, Y \in \mathcal{T}^1(M)$, define

$$2 \cdot \omega_{X,Y}(Z) = X g(Y, Z) + Y g(X, Z) - Z g(X, Y) - g(Y, [X, Z]) - g(X, [Y, Z]) + g(Z, [X, Y])$$

Claim: $\omega_{X,Y} \in \mathcal{T}_1(M)$ $(\Longleftrightarrow \omega_{X,Y}(f \cdot Z) = f \cdot \omega_{X,Y}(Z))$

Pf: Follows from the fact that

$$[X, fZ] = X(f) \cdot Z + f \cdot [X, Z].$$
$$[Y, fZ] = Y(f) \cdot Z + f \cdot [Y, Z].$$

$$X g(Y, fZ) = X(f)\cdot g(Y,Z) + f\cdot X g(Y,Z)$$
$$Y g(X, fZ) = Y(f)\cdot g(X,Z) + f\cdot Y g(X,Z)$$

and a simple calculation

We then define $\nabla_X Y \equiv \omega^{\#}_{X,Y}$. Note that ∇ satisfies the Koszul formula (as it must)

In particular if $f \in C^\infty(M)$, then

$$\begin{aligned}2g(\nabla_X fY, Z) &= X(f)\cdot g(Y,Z) + f X g(Y,Z) \\ &\quad + f Y g(X,Z) - \cancel{Z(f) g(X,Y)} - f Z g(X,Y \\ &\quad - f g(Y, [X,Z]) - f g(X, [Y,Z]) \\ &\quad + \cancel{Z(f) g(X,Y)} + X(f) g(Z,Y) + f g(Z, [X, \\ &= 2X(f) g(Y,Z) + 2f g(\nabla_X Y, Z)\end{aligned}$$

So $2g(\nabla_X f\cdot Y - X(f) Y - f \nabla_X Y, Z) = 0 \quad \forall Z$

$$\Rightarrow \quad \nabla_X (fY) = X(f) Y + f \nabla_X Y$$

Properties C1, C2 are similarly proved. So ∇ defines a linear connection on M.

Metric compatibility & symmetry are also similarly verified using Koszul formula.

Rk. The Christoffel symbols of a Levi-Civita connection are denoted by $\Gamma_{ij}^{\;k} := \langle dx^k, \nabla_{\partial_i} \partial_j \rangle$

i.e
$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^{\;k}\, \partial_k .$$

Symmetry $\iff$ $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$ (since $[\partial_i, \partial_j] = 0$).

Applying the Koszul formula to $X = \partial_i$, $Y = \partial_j$, $Z = \partial_l$, and setting $\partial_i g_{jl} = g_{jl;i}$ etc,

$$2\Gamma_{ij}^{k}\, g_{kl} = g_{jl;i} + g_{il;j} - g_{ij;l}.$$

So

$$\boxed{\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(g_{jl;i} + g_{il;j} - g_{ij;l} \right)}$$

<u>Example</u>: On $\mathbb{R}^n$, we have the trivial connection

$$\overline{\nabla}_X Y = X^i \partial_i Y^j \cdot \hat{e}_j$$

where $\{\hat{e}_j\}$ are the st. basis vector of $\mathbb{R}^n$.
Clearly $\overline{\nabla}$ is $g_{\mathbb{R}^n}$-compatible & symmetric.
So $\overline{\nabla}$ is <u>the</u> Levi-Civita connection on $(\mathbb{R}^n, g_{\mathbb{R}^n})$.

<u>Prop 6.2</u>: Let (M, g) be a Riemannian sub-mfld of (N, h). Let ∇_N, ∇_M be the Levi-Civita connections on M & N. Let X, Y be vector fields on M & let $\tilde{X}, \tilde{Y}$ be <u>any</u> extensions to vector fields on N. Then for any $p \in M$,

$$((\nabla_M)_X Y)_p = ((\nabla_N)_{\tilde{X}} \tilde{Y})^T_p$$

where for any $v \in T_pN$, v^T is the orthogonal projection to T_pM.

Pf: Define $(\nabla^T)_X Y = ((\nabla_N)_{\tilde{X}} \tilde{Y})^T$. We have seen earlier that ∇^T is a connection on M (in fact this works for <u>any</u> projection, not neccessarily orthogonal).

Claim 1 ∇^T is symmetric

Pf:
$$T_{\nabla^T}(X,Y) := (\nabla^T)_X Y - \nabla^T_Y X - [X,Y]$$
$$= ((\nabla_N)_{\tilde{X}}\tilde{Y} - (\nabla_N)_{\tilde{Y}}\tilde{X} - [\tilde{X},\tilde{Y}])^T$$
$$= 0 \text{ since } [X,Y] = [\tilde{X},\tilde{Y}] \text{ \& } \nabla_N \text{ is symmetric.}$$

Claim 2: $\nabla^T g \equiv 0$

Pf: Let $X, Y, Z \in \mathcal{T}^1(M)$ & $\tilde{X}, \tilde{Y}, \tilde{Z}$ their extension

At $p \in M$,
$$Z g(X,Y) - g(\nabla^T_Z X, Y) - g(X, \nabla^T_Z Y)$$
$$= \tilde{Z} h(\tilde{X},\tilde{Y}) - g(((\nabla_N)_{\tilde{Z}}\tilde{X})^T, Y) - g(X, ((\nabla_N)_{\tilde{Z}}\tilde{Y})^T)$$

At p, since ^{T} is an orthogonal projection,
$$g_p(((\nabla_N)_{\tilde{Z}}\tilde{X})^T_p, Y_p) = h_p((\nabla_N)_{\tilde{Z}_p}\tilde{X}, \tilde{Y})$$
$$g_p(X_p, ((\nabla_N)_{\tilde{Z}}\tilde{Y})^T_p) = h(\tilde{X}, (\nabla_N)_{\tilde{Z}}\tilde{Y})$$

So
$$\nabla_Z g(X,Y) = (\nabla_N)_{\tilde{Z}} h(\tilde{X},\tilde{Y}) = 0$$
since ∇_N is h-compatible.

So ∇^T is the Levi-Civita connection on (M,g)

Uniqueness $\Longrightarrow$ $\nabla^T \equiv \nabla_M$.

<u>Prop 6.3</u> (Naturality) Sps $\varphi : (M, g) \longrightarrow (\tilde{M}, \tilde{g})$ is an isometry. If ∇ and $\tilde{\nabla}$ are the resp. Levi-Civita connections, then for any $X, Y \in \mathcal{T}^1(M)$

$$\varphi_*(\nabla_X Y) = \tilde{\nabla}_{\varphi_* X} \varphi_* Y.$$

In particular if $\gamma : I \to M$ is a C^1-curve & X is a v.f along γ, then

$$\varphi_* D_t X = \tilde{D}_t(\varphi_* X).$$

<u>Pf</u>: Check that $\varphi_*^{-1} \tilde{\nabla}_{\varphi_* X} \varphi_* Y$ is a Levi-Civita connection on M, and hence must be ∇

- <u>PARALLEL TRANSPORT WITH LEVI-CIVITA CONNECTIO</u>

<u>Prop</u> 6.4 Let (M, g) be a Riemannian mfld, & ∇ be the Levi-Civita connection. If $\gamma : I \to \mathbb{R}$ is a C^1-curve, then $P_{\gamma; t_0, t_1} : T_{\gamma(t_0)} M \longrightarrow T_{\gamma(t_1)} M$ the $\parallel$-transport map, is an <u>isometry</u>.

<u>Pf</u>: Let $v, w \in T_{\gamma(t_0)} M$ & $v(t) = P_{\gamma; t_0, t} v$, $w(t) = P_{\gamma; t_0, t} w$. Since $\nabla g \equiv 0 \Longrightarrow$

$$\boxed{\frac{d}{dt} \langle v(t), w(t) \rangle = \langle D_t v, w \rangle + \langle v, D_t w \rangle}$$

Since $v(t), w(t)$ are $\parallel$-along γ, $D_t v, D_t w = 0$.

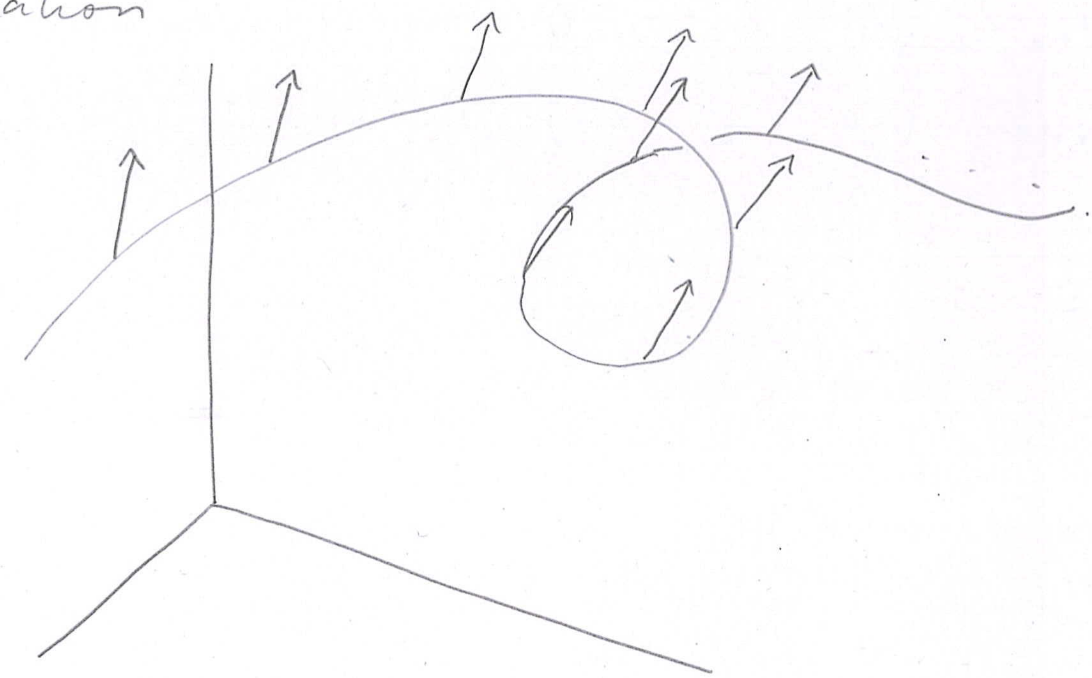
So $\langle v(t), w(t) \rangle =$ constant. In particular.

$$\langle P_{\gamma, t_0, t_1} v, P_{\gamma, t_0, t_1} w \rangle_{\gamma(t_1)} = \langle v, w \rangle_{\gamma(t_0)}.$$

Examples 1) **∥- transport in $\mathbb{R}^n$**: Let $\gamma(t)$ be a C^1-curve in $\mathbb{R}^n$ & $\vec{v} \in T_{\gamma(t_0)}\mathbb{R}^n \cong \mathbb{R}^n$. If $\vec{v}(t) = P_{\gamma, t_0, t} v$ then since $\Gamma_{ij}^k \equiv 0$, & $\vec{v}(t)$ is $\parallel \Longleftrightarrow$

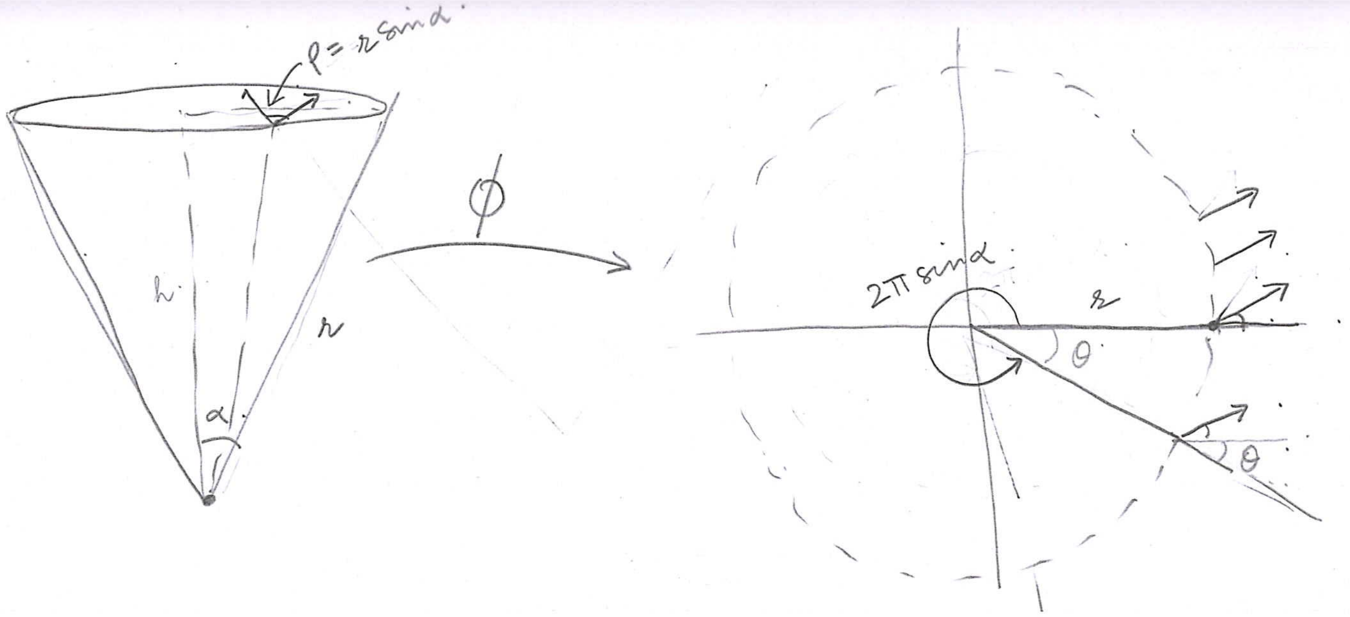
$$\dot{V}^k(t) = 0.$$

So $V^k(t) = V^k \ \forall\ t$. i.e ∥- transport in $\mathbb{R}^n$ is translation



2) **∥- transport around cone** For $\alpha \in (0, \pi/2)$, define

$$C_\alpha = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \tan^2\alpha,\ z > 0\}$$



The $C_\alpha \setminus \{(0,0,0)\}$ is isometric to a sector in $\mathbb{R}^2$ of angle $2\pi \sin\alpha$.

Let $\gamma : [0,1] \to C_\alpha$ be a circle $z = h$ on C_α. If $v \in T_{\gamma(0)}C_\alpha$. Then, one can compute $\parallel$-transport by transporting the problem to $\mathbb{R}^2$. If v makes angle β w.r.t the radial line, then $P_{\gamma;0,1}(v)$ make the angle $\beta - 2\pi\theta$ (see figure) So angle between v & $P_{\gamma,0,1}(v)$ is $2\pi\theta = 2\pi(1 - \sin\alpha)$.