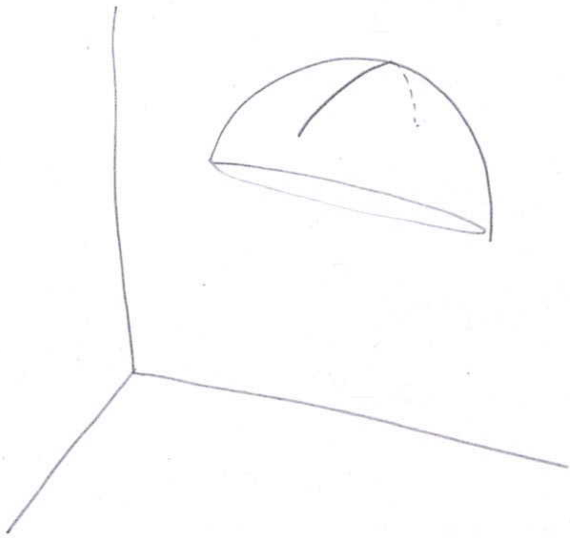
LECTURE - 7

· GEODESICS. Let (M, g) be a Riemannian mfld.

Example: Sps $M \subset \mathbb{R}^3$ w/ the induced metric g. Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ be a curve.



Intuitively, we want geodesics to be length minimizing. A neccessary condition seems to be $\ddot{\gamma}(t) \perp T_{\gamma(t)} M$. Note that if $\bar{D}_t$ & D_t are the covariant derivative along γ, then $\ddot{\gamma}(t) = D_t \dot{\gamma}(t)$ and so $\ddot{\gamma}(t) \perp T_{\gamma(t)} M \iff D_t \dot{\gamma}(t) = 0 \;\; \forall t$.

Defn: A smooth regular parametrized curve $\gamma : I \to M$ is called a geodesic if $D_t \dot{\gamma}(t) = 0 \;\; \forall t \in I$.

Rk 1). If $M \subset \mathbb{R}^n$ w/ induced metric, then

γ is a geodesic $\iff \ddot{\gamma} \perp T_{\gamma(t)} M$.

2) If γ is a geodesic, then

$$\frac{d}{dt} |\dot{\gamma}(t)|^2 = 2 \langle D_t \dot{\gamma}, \dot{\gamma} \rangle = 0$$

So, $|\dot{\gamma}(t)| = \text{const}$.

If we reparametrize s.t $|\dot{r}(t)| = 1$, we call r a normal geodesic.

3) If $\tilde{r}$ is a re-parametrization of a geodesic r i.e $\tilde{r}(s) = r(t(s))$ for some monotonic function $t(s)$, then $\tilde{r}'(s) = \dot{r}(t(s)) \cdot t'(s)$ So

$$D_s \tilde{r} = \nabla_{\tilde{r}'(s)} \tilde{r}'(s)$$

$$= \dot{r}(t(s)) \cdot t''(s) + [t'(s)]^2 \cdot D_t \dot{r}(t(s))$$

$$= \dot{r}(t(s)) \cdot t''(s)$$

So $\tilde{r}$ is a geodesic $\iff t''(s) = 0$ or $t(s) = as+b$.

Examples 1) Geodesics in $\mathbb{R}^n$: $\bar{D}_t \dot{r} = \ddot{r}(t)$. So

$$\bar{D}_t r = 0 \iff \ddot{r}(t) = 0 \iff r(t) = \vec{u}t + \vec{a}$$

So geodesics are straight lines

2) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

$$r(t) = (\cos t, \sin t, t)$$

Then $\ddot{r}(t) = (-\cos t, -\sin t, 0)$.

The outward normal to the cylinder at (x, y, z) is given by $\vec{n}(x, y, z) = (x, y, 0)$.

Clearly $\ddot{r} \parallel \vec{n}(r(t))$ So r is a geodesic.

3) Let $M \subset \mathbb{R}^3$ & Π a plane s.t $\Gamma \in M \cap \Pi$ is a smooth curve s.t $\forall p \in \Gamma$, Π contains the normal to M at p.

If we parametrize Γ by $r(t)$ s.t $|\dot{r}(t)| = \text{const}$. Then $r(t)$ is a geodesic. To see this, $|\dot{r}(t)| = \text{const}$ $\Rightarrow$ $\ddot{r} \perp \dot{r}$. Also $\ddot{r}(t) \in \Pi$. But Π is spanned by normal $\hat{n}$ & $\dot{r}(t)$. So $\ddot{r}(t) \perp T_{r(t)}M$.

e.g: Great circles in S^2 are geodesics

· Existence & Uniqueness Of Geodesics

We need the foll general result on 2^{nd} order ODEs.

Th^m 7.1 Let $U, V \subset_{open} \mathbb{R}^n$ & $F: U \times V \longrightarrow \mathbb{R}^n$ be smooth. Consider

$$\frac{d^2\vec{x}}{dt^2} = F\left(\vec{x}, \frac{d\vec{x}}{dt}\right). \qquad (*)$$

For each $(\vec{x}_0, \vec{v}_0) \in (U, V)$. $\exists$ nbd, $U_0 \times V_0$ & $\varepsilon > 0$ s.t $\forall$ $(\vec{x}, \vec{v}) \in U_0 \times V_0$ the equation has a unique solution $\vec{x}_{\vec{v}} : (-\varepsilon, \varepsilon) \longrightarrow U$ w/ initial condition $\vec{x}_{\vec{v}}(0) = \vec{x}$ & $\vec{x}_{\vec{v}}'(0) = \vec{v}$. Moreover the map $X : U_0 \times V_0 \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$, defined by $X(\vec{x}, \vec{v}, t) = \vec{x}_v(t)$ is smooth

Rk: The key point is that ε is independent of $(\vec{x}, \vec{v})$.

Th^m 7.2 Let $p \in M$. $\exists$ nbd U_0 of p & real numbers $\delta > 0$ & $\varepsilon > 0$ s.t $\forall$ $x \in U_0$ & $\forall$ $v \in T_xM$ s.t $|v| < \delta$, $\exists\,!$ geodesic $r_v : (-\varepsilon, \varepsilon) \longrightarrow M$ satisfying $r_v(0) = x$ & $r_v'(0) = v$

Cor 7.3 For any $p \in M$ & $v \in T_pM$, $\exists\ \varepsilon > 0$ & a unique geodesic $\gamma_v : (-\varepsilon, \varepsilon) \longrightarrow M$ s.t. $\gamma_v(0) = p$ & $\gamma_v'(0) = v$.

Pf: (Rescaling trick) Let $\tilde{v} = c^{-1} v$, where $c = \frac{2|v|}{\delta} \Rightarrow |\tilde{v}| < \delta$.

So Th^m 7.2 $\Rightarrow$ $\exists\ \varepsilon_0$ & a! geodesic $\gamma_{\tilde{v}}$ s.t

$\gamma_{\tilde{v}}(0) = p$ & $\gamma_{\tilde{v}}'(0) = \tilde{v}$ Consider

$$\gamma_v(t) = \gamma_{\tilde{v}}(ct).$$

Then since this is a linear change of parametrization, γ_v is a geodesic. Moreover

$$\gamma_v'(0) = c \cdot \gamma_{\tilde{v}}'(0) = c\tilde{v} = v.$$

Pf of Th^m 7.2 Let (U, φ) be a coordinate chart near p. Then $TM|_{U_0} \approx U_0 \times \mathbb{R}^n$ We can assume $\varphi(p) = 0$. Sps $\{x^i\}$ are coordinates on U Then $\gamma : I \rightarrow U$ is a geodesic $\Longleftrightarrow$ $\gamma(t) = (x^1(t), \ldots x^n(t))$ satisfy

$$\ddot{x}^k(t) + \Gamma^k_{ij}(x(t)) \cdot \dot{x}^i(t)\, \dot{x}^j(t) = 0 \quad (**)$$

Applying Th^m 7.1 to $(x_0, \vec{v}_0) = (0, \vec{0}) \in U \times \mathbb{R}^n$

$\exists$ nbd U_0 of p & V_0 of $\vec{0}$; $\varepsilon > 0$ s.t $\forall$ $(\vec{x}, \vec{v}) \in \varphi(U_0) \times V_0$, $\exists!$ $\vec{x}_v : (-\varepsilon, \varepsilon) \longrightarrow U$ solving $(*)$ & satisfying $\vec{x}_v(0) = \vec{x}$, $\vec{x}_v'(0) = \vec{v}$

Choose δ small s.t if $|v| < \delta$, $v \in T_xM$, $x \in U_0$ then $v = \sum v^i \partial_i$ with $\vec{v} = (v^1, \dots, v^n) \in V_0$.

Then $\forall\, x \in U_0$, $\forall\, v \in T_xM$ s.t $|v| < \delta$,

$$\gamma_v(t) := \varphi^{-1}(\vec{x}_v(t)).$$

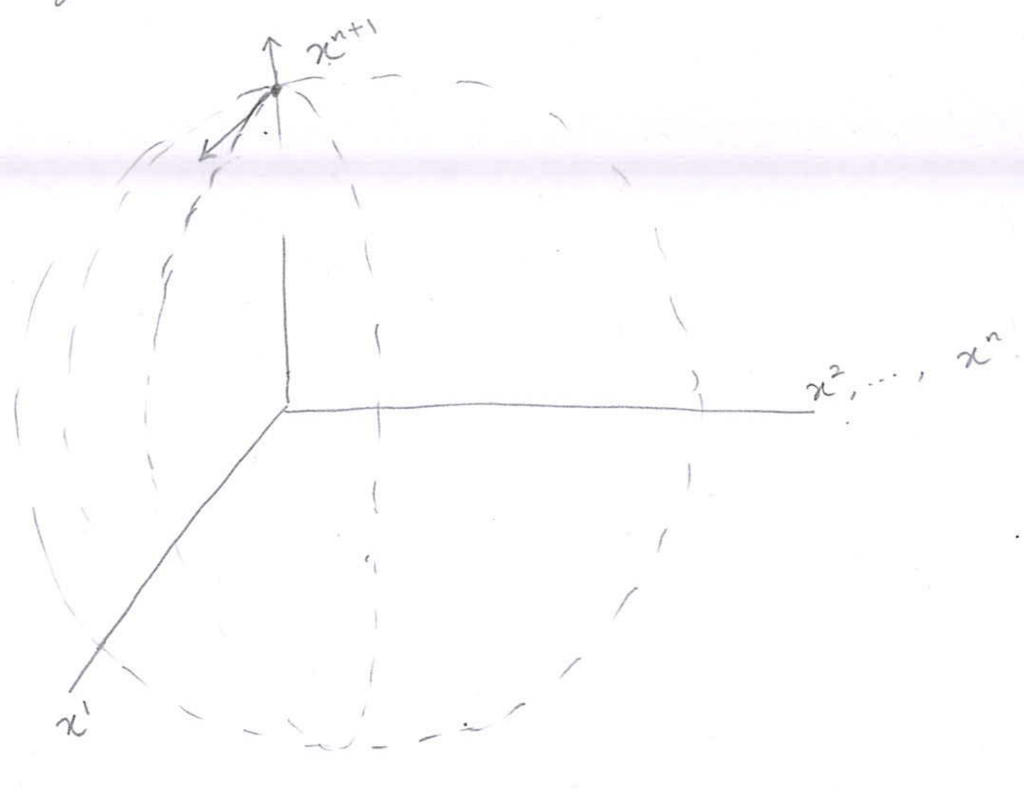
is a geodesic, & $\gamma_v(0) = x$, $\gamma_v'(0) = v$.

Rk Follows from the uniqueness part, that $\forall\, p \in M$ & $v \in T_pM$, $\exists$ a unique, maximal geodesic $\gamma : I \longrightarrow M$ w/ $\gamma(0) = p$, $\gamma'(0) = v$. Simply let I be the union of all open intervals on which such a geodesic is defined. We call γ the geodesic at p in the direction of v & write it as γ_v.

· Geodesics On Spheres

Th^m 7.4 The geodesics on $\mathbb{S}^n_R$ are precisely the "great circles" (intersections of $\mathbb{S}^n_R$ w/ 2-planes through the origin), w/ constant speed parametrizations.

Pf:



<u>Step 1</u> Sps geodesic $\gamma(t) = (x^1(t), \cdots, x^{n+1}(t))$ starts at the north pole N & $\gamma'(0) = c \cdot \partial_1$.

<u>Claim</u> $x^2(t) = \cdots = x^n(t) = 0$.

<u>Pf</u>: Sps not. Then $\exists$ t_0 s.t. $x^i(t_0) \neq 0$ for some $i = 2, \cdots, n$

Let $\varphi: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$, $\varphi(x) = (x^0, \cdots -x^i, \cdots, x^n)$. Then $\varphi|_{S^n_R}$ is an isometry s.t $\varphi(N) = N$ & $d\varphi_N(V) = V$.

Uniqueness of geodesics $\Longrightarrow$ $\varphi(\gamma) = \gamma$. But $\varphi(\gamma(t_0)) \neq \gamma(t_0)$ in the i^{th} coordinate.

<u>Step 2</u>: For any other $\vec{p} \in S^n_R$, $\exists$ a $A \in O(n+1)$. s.t $A\vec{p} = N$. A takes 2 planes through origin to 2-planes through origin & we are done.

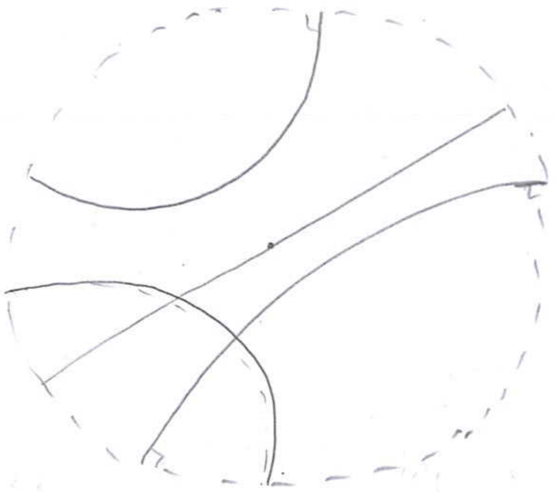
• <u>GEODESICS IN HYPERBOLIC SPACE</u>

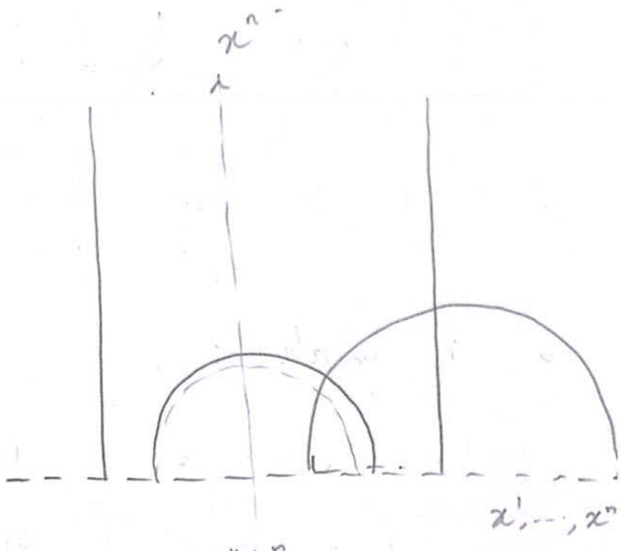
<u>Th^m 7.5</u>: The geodesics in hyperbolic sp. are the following curves w/ unit speed parametrization

1) <u>Hyperboloid model $\mathbb{H}^n_R$</u>: The "great hyperbolas", or the intersections of $\mathbb{H}^n_R$ w/ 2-planes through origin.

2) <u>Ball Model $\mathbb{B}^n_R$</u>: Line segments through the origin & circular arcs that intersect $\partial \mathbb{B}^n_R$ orthogonally.

3) <u>Upper Half space $\mathbb{U}^n_R$</u> Verticle half lines & semi-circles w/ center on $x^n = 0$.





Pf" 1) Recall $\mathbb{H}^n_R := \{(x^0, \cdots, x^n) \mid \langle x, x \rangle := -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -R^2,\ x^0 > 0\}$.

The Lorenz - group is defined by

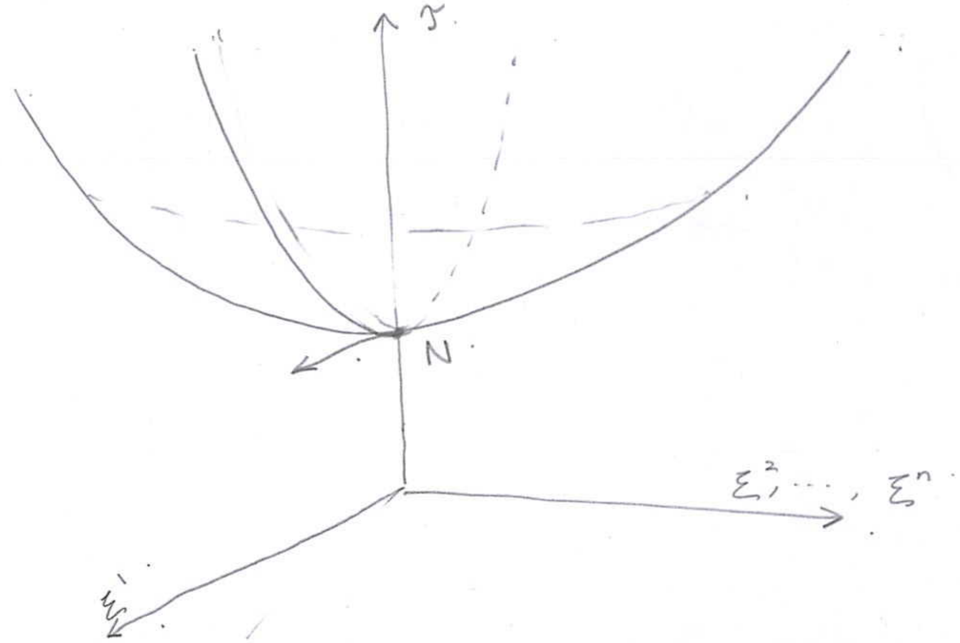
$$O(n,1) = \{A \in GL(n+1, \mathbb{R}) \mid A \text{ preserves } \langle\, , \rangle\}.$$

$$O^+(n,1) = \{A \in O(n,1) \mid A(\mathbb{H}^n_R) = \mathbb{H}^n_R\}$$

Then $O^+(n,1) \subseteq O(n,1)$ is of index 2 (if $A \in O(n,1)$ then either $A(\mathbb{H}^n_R) = \pm \mathbb{H}^n_R$)

FACT: $O^+(n,1) = \mathrm{Isom}(\mathbb{H}^n_R)$ & in fact $O^+(n,1)$ acts transitively on $\mathbb{H}^n_R$

So to calculate geodesics, one can simply look at $N = (R, 0, \cdots, 0)$.



Consider a geodesic w/ $\gamma(0) = N$, $\gamma'(0) = V_1 = c \cdot \frac{\partial}{\partial \xi^1}$.
Again by symmetry, γ lies entirely in the $\tau - \xi^1$ plane. & is in fact the intersection of $\mathbb{H}^n_R$ w/ $\tau - \xi^1$ plane. Since for any other $x \in \mathbb{H}^n_R$ & any $\vec{V} \in T_x \mathbb{H}^n_R$, $\exists A \in O(n,1)$ s.t $Ax = N$, & $A\vec{V} = \vec{V}_1$ all geodesics must be intersection of $\mathbb{H}^n_R$ w/ 2-plane passing through origin.

(2) & (3) can then be proved using isometries w/ $\mathbb{H}^n_R$

· <u>Discussion of geodesics in $\mathbb{U}^2$</u> (i.e $R = 1$).

Then $$g_{\mathbb{U}^2} = \frac{dx^2 + dy^2}{y^2}$$

One can compute: $\Gamma^x_{xx} = \Gamma^x_{yy} = \Gamma^y_{xy} = \Gamma^y_{yx} = 0$.
$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\Gamma^y_{xx} = \Gamma^y_{yy} = -1/y.$$

Consider $\gamma_0(t) = (0, e^t)$, $t \in (-\infty, \infty)$. Then

$$|\gamma_0'(t)|_{g_{\mathbb{H}^2}} = 1.$$

Geodesic eq reduce to

$$\left.\begin{aligned} \ddot{x}(t) - \frac{2}{y}\,\dot{x}\,\dot{y} &= 0 \\ \ddot{y} + \frac{(\dot{x})^2 - (\dot{y})^2}{y} &= 0 \end{aligned}\right\} \quad (***)$$

Clearly γ_0 satisfies the geodesic equation.

<u>FACT</u>: $PSL(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$ & acts transitively by Mobius transformation & takes γ_0 to vertical lines (translation) or semi-circles.

<u>A more direct arg</u>: $(***)$ can be written as

$$\left(\frac{\dot{x}}{y^2}\right)' = 0$$

$$\left(\frac{\dot{y}}{y^2}\right)' = -\frac{(\dot{x})^2 + (\dot{y})^2}{y^3}$$

So $\dot{x} = Cy^2$. If we impose $\gamma(t) = (x(t), y(t))$ has unit speed, then $(\dot{x})^2 + (\dot{y})^2 = y^2$. So 2^{nd} eq is

$$\frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = -\frac{1}{y} \qquad (3)$$

Note $\dfrac{d}{dt} = \dfrac{d}{dx}\cdot \dot{x} = Cy^2 \cdot \dfrac{d}{dx}$, so $\dot{y} = \dfrac{dy}{dx}\cdot Cy^2$

So, $(3) \Longleftrightarrow$

$$C^2 y^2 \frac{d^2y}{dx^2} = -\frac{1}{y}.$$

or $$\frac{d^2y}{dx^2} = -R^2 y^{-3}, \quad R = C^{-1}, \; C \neq 0.$$

Solving this we get $y^2 + (x-a)^2 = R^2$.

$C = 0$ gives vertical st. lines.