LECTURE - 8

• **Recall**: $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ is geodesic if $\nabla_{\dot\gamma} \dot\gamma \equiv 0$.

Thm 7.2: For every $p_0 \in M$, $\exists$ a nbd U_0 of p_0 & $\varepsilon, \delta > 0$ s.t $\forall$ $p \in U_0$, $\forall$ $v \in T_pM$, $|v| < \delta$, $\exists$ a unique geodesic $\gamma_v : (-\varepsilon, \varepsilon) \longrightarrow M$ s.t $\gamma_v(0) = p$ & $\gamma_v'(0) = v$.

Cor 8.1: For every $p_0 \in M$, $\exists$ a nbd U_0 of p_0 & $\varepsilon_0 > 0$ s.t $\forall$ $p \in U_0$, $\forall$ $v \in T_pM$, $|v| < \varepsilon_0$, $\exists$ a unique geodesic $\gamma_v : (-2, 2) \longrightarrow M$ s.t. $\gamma_v(0) = p$ & $\gamma_v'(0) = v$. Moreover $\Gamma : TU_0 \times (-2,2) \to M$, $\Gamma(v,t) = \gamma_v(t)$ is C^∞

Pf (Rescaling trick). Let U_0, ε, δ be as above, and define $\varepsilon_0 = \varepsilon\delta/2$. Then if $v \in T_pM$, $p \in U_0$ s.t. $|v| < \varepsilon_0$, then clearly $|2v/\varepsilon| < \delta$. Define

$$\gamma_v(t) := \gamma_{2v/\varepsilon}(\varepsilon t/2).$$

Thm 7.2 $\Rightarrow$ $\gamma_v : (-2, 2) \longrightarrow M$, and is a geodesic. Since $\gamma_{2v/\varepsilon}(t')$ is geodesic & $t' = \varepsilon t/2$ is a linear change of coordinates. Also, $\gamma_v(0) = \gamma_{2v/\varepsilon}(0) = p$.

$$\gamma_v'(0) = \frac{\varepsilon}{2} \cdot \gamma_{2v/\varepsilon}'(0) = \frac{\varepsilon}{2} \cdot \frac{2v}{\varepsilon} = v.$$

Done!

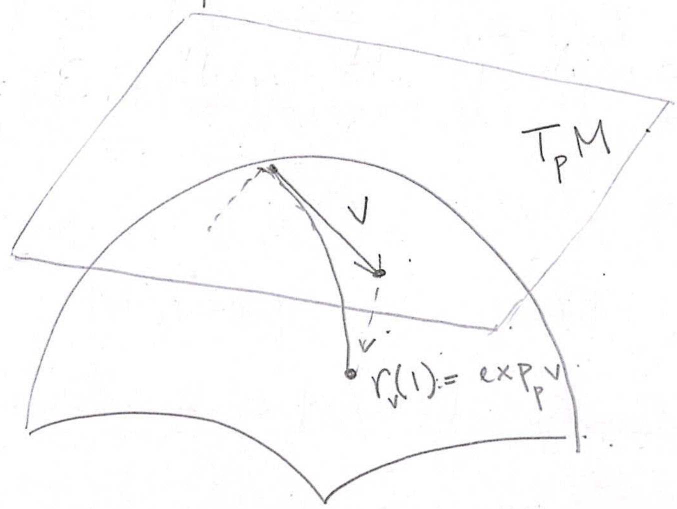
<u>Def^n</u>: The <u>domain</u> of the <u>exponential map</u> is defined to be

$$\mathcal{E} = \{ v \in TM \mid \gamma_v \text{ is defined on some interval containing } [0,1] \},$$

and then the <u>exponential</u> <u>map</u> $\exp : \mathcal{E} \longrightarrow M$ is defined by

$$\exp(v) := \gamma_v(0).$$

For each $p \in M$, we denote by $\exp_p$, the restriction of the map to $\mathcal{E}_p := \mathcal{E} \cap T_pM$.



<u>Prop 8.2</u>: (a) $\mathcal{E}$ is an <u>open</u> subset of TM containing the zero section & $\mathcal{E}_p$ is <u>star shaped</u> w.r.t $0 \in T_pM$, i.e if $v \in \mathcal{E}_p$, then $tv \in \mathcal{E}_p \ \forall \ t \in [0,1]$.

b) $\forall \ v \in T_pM$, $\gamma_v(t) = \exp_p(tv) \ \forall \ t$ s.t $tv \in \mathcal{E}_p$

(c) $\exp : \mathcal{E} \longrightarrow M$ is smooth.

<u>Pf</u>: Lee Prop 5.7.

<u>Th^m 8.3</u>: For any $p \in M$, $\phi : \mathcal{E} \longrightarrow M \times M$ defined by

$$\phi(v) = (\pi(v), \exp_{\pi(v)}(v))$$

is a local diffeomorphism from a nbd W_p of 0_p in TM onto a nbd of $(p,p) \in M \times M$.

<u>Pf</u>: We use the inverse function. Let $(\mathcal{U}, \varphi)$ be a chart in a nbd of p w/ coordinates $(x^1, \cdots, x^n)$.

Then $TM|_{\mathcal{U}} \approx \mathcal{U} \times \mathbb{R}^n$. In fact

$$(x^1, \cdots, x^n, t^1, \cdots, t^n) \longrightarrow t^i \frac{\partial}{\partial x^i}$$

Clearly $T_{(p,\vec{0})} TM = \operatorname{sp}\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial t^1}, \cdots, \frac{\partial}{\partial t^n}\right\}$

We also denote the induced coordinates on $\mathcal{U} \times \mathcal{U} \subseteq M \times M$ by $(x^1_{(1)}, \cdots, x^n_{(1)}, x^1_{(2)}, \cdots, x^n_{(2)})$ i.e. for any $(q_1, q_2) \in \mathcal{U} \times \mathcal{U}$,

$$x^i_{(1)}(q_1, q_2) = x^i(q_1)$$
$$x^i_{(2)}(q_1, q_2) = x^i(q_2)$$

<u>Claim</u>: In the coordinate system above

$$D_{(p,\vec{0})}\phi = \begin{pmatrix} id & id \\ 0 & id \end{pmatrix}.$$

<u>Pf</u>: In these coordinates

$$\phi(x, \vec{v}) = (x, \exp_x \vec{v})$$

Note that if $\vec{v} = \vec{0}$, then $\phi(x, \vec{0}) = (x, x)$.

and so, $D_{(p,\vec{0})}\phi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i_{(1)}} + \frac{\partial}{\partial x^i_{(2)}}$

Similarly if $x = p$, then $\phi(p, \vec{v}) = (p, \exp_p \vec{v})$.

$$D_{(p,\vec{0})}\phi\left(\frac{\partial}{\partial t^j}\right) = \frac{d}{ds}\bigg|_{s=0} \phi(p, s\hat{e}_j) = \frac{d}{ds}\bigg|_{s=0}\left(p, \exp_p s\frac{\partial}{\partial x^j}\right).$$

$$= \frac{d}{ds}\bigg|_{s=0} \left(p, \gamma_{\frac{\partial}{\partial x^j}}(s)\right)$$

$$= \left(0, \gamma'_{\frac{\partial}{\partial x^j}}(0)\right)$$

$$= \frac{\partial}{\partial x^j_{(2)}}$$

So $D_{(p,\vec{0})}\phi = \begin{pmatrix} id & id \\ 0 & id \end{pmatrix}$.

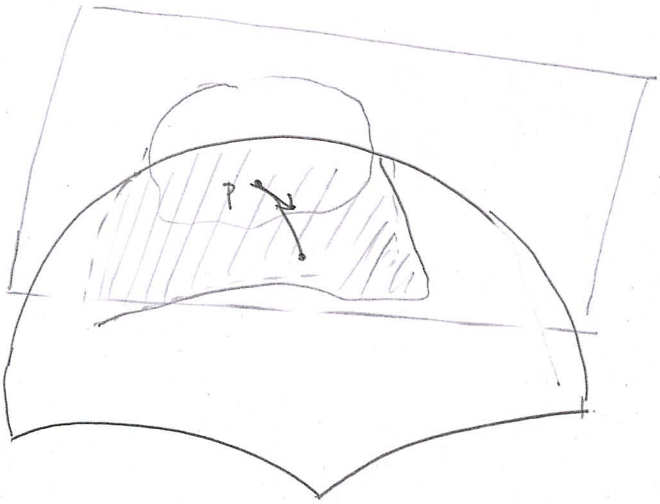
By inverse function theorem, the proof is complete since $\det D_{p,\vec{0}}\phi = 1 \neq 0$.

(of the proof).

<u>Cor 8.4</u> Let $p \in M$. Then $\exists$ open nbd $0 \in \mathcal{V} \subset T_pM$ & $p \in \mathcal{U} \subset M$ of p s.t. $\exp_p : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism.

<u>Rk</u>: More invariantly if we identify $T_0T_pM \cong T_pM$, then

$$(\exp_p)_*(v) = \frac{d}{ds}\bigg|_{s=0} \exp_p(sv) = \frac{d}{ds}\bigg|_{s=0} \gamma_v(s) = \gamma_v'(0) = v.$$



• Normal Coordinates

Defⁿ. 1) A neighbourhood $\mathcal{U}$ of $p \in M$ that is the diffeomorphic image of a star shaped open nbd of $0 \in T_pM$ under $\exp_p$ is called a normal neighbourhood.

2). A normal coordinate chart on a normal nbd $\mathcal{U}$ is a map $\varphi := E^{-1} \circ \exp_p^{-1} : \mathcal{U} \longrightarrow \mathbb{R}^n$, where $E : \mathbb{R}^n \longrightarrow T_pM$ is a isomorphism given by an orthonormal (w.r.t 'g') basis $(E_1, \ldots, E_n)$ of T_pM i.e $E(e_i) = E_i$. The corresponding set of coordinates are called normal coordinates.

3). Let $\{x^i\}$ be normal coordinates on $\mathcal{U}$ centered at $p \in M$. For any $x \in \mathcal{U}$, the radial distance is defined by

$$r(x) = \left(\sum (x^i)^2 \right)^{1/2}$$

& the unit radial vector field on $\mathcal{U}$ is defined by

$$\frac{\partial}{\partial r} := \frac{x^i}{r} \frac{\partial}{\partial x^i}$$

Prop 8.5: Let $(\mathcal{U}, \varphi, (x^i))$ be any normal coordinate chart centered at p

(a) $\varphi(p) = (0, 0, \dots, 0)$.

(b) For any $V = V^i \partial_i \in T_pM$,

$$\varphi \circ \gamma_V(t) = (tV^1, \dots, tV^n).$$

as long as $\gamma_V(t) \in \mathcal{U}$

(c). $\forall\ q \in \mathcal{U} - p$; $\partial/\partial r|_q$ is the velocity vector of the unit speed geodesic from p to q.

(d) $g_{ij}(p) = \delta_{ij}$, $g_{ij,k}(p) = 0$, $\Gamma_{ij}^k(p) = 0\ \forall\ i, j, k$.

Pf: (a) $\varphi(p) = E^{-1} \circ \exp_p^{-1}(p)$

$$= E^{-1}(\vec{0}_p) = \vec{0}$$

(b). Recall that $\gamma_V(t) = \gamma_{tV}(1) = \exp_p(tV)$

So

$$\varphi \circ \gamma_V(t) = E^{-1} \circ \exp_p^{-1} \exp_p(tV)$$

$$= E^{-1}(tV) = tE^{-1}(V)$$

But $E_i = \partial_i$ (since $E(x^1, \dots, x^n) = x^i E_i$)

So, $\varphi \circ \gamma_V(t) = t\, E^{-1}(V^i \partial_i)$

$$= t \cdot V^i e_i$$

$$= (tV^1, \ldots, tV^n).$$

(c). Let $V \in T_pM$ s.t. $q = \exp_p V$. Let $\gamma(t) := \exp_p(t\,V/|V|)$. Then $|\gamma'(0)| = 1$. Since γ is a geodesic, $|\gamma'(t)| = 0$ $\forall\, t \in [0, |V|]$. Also $\gamma(|V|) = q$. Sps $V = V^i \partial_i$. Since ∂_i are orthonormal, $|V| = \left(\sum (V^i)^2\right)^{1/2}$.

By (b),

$$\gamma(t) = \gamma_{V/|V|}(t) = \left(\frac{tV^1}{|V|}, \ldots, \frac{tV^n}{|V|}\right).$$

So $$\gamma'(t) = \frac{V^i}{|V|} \cdot \frac{\partial}{\partial x^i}$$

By defn $\varphi(q) = E^{-1} \circ \exp_p^{-1}(q)$

$$= E^{-1}(V^i \partial_i)$$

$$= V^i e_i$$

So $r(q) = \left(\sum (V^i)^2\right)^{1/2} = |V|$ & hence.

$$\gamma'(t) = \left.\frac{\partial}{\partial r}\right|_q$$

(d). Since $\{\partial_i\}$ is orthonormal at p, $g_{ij}(p) = \delta_{ij}$. Now, let $V = V^i \partial_i \in T_pM$. By (b) $\gamma_V(t) = (tV^1, \ldots, tV^n)$

From the geodesic equation, since $\ddot{\gamma}_V^k = 0$,

$$\Gamma_{ij}^k(\gamma_V(t))\, v^i v^j = 0 .$$

Setting $t=0$, $\Gamma_{ij}^k(p)\, v^i v^j = 0 \quad \forall\, k$.

Since this is true $\forall\ V \in T_pM$, $\Gamma_{ij}^k(p) = 0 \ \forall\, i,j,k$

But, then by metric compatibility,

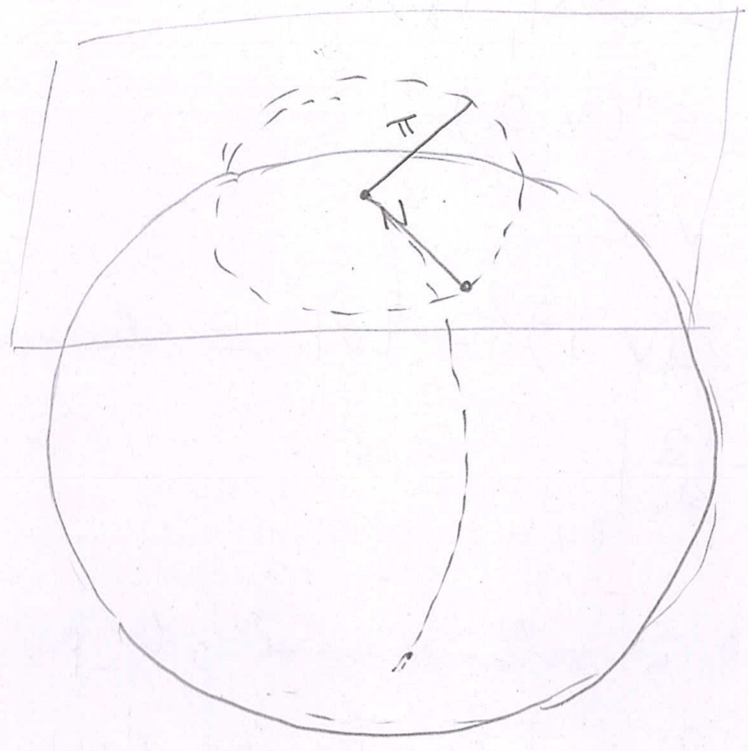
$$g_{ij,k}(p) = \partial_k g(\partial_i, \partial_j) = g(\nabla_{\partial_k}\partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k}\partial_j)$$

$$= g(\Gamma_{ki}^l(p)\partial_l, \partial_j) + g(\partial_i, \Gamma_{kj}^l(p)\partial_l)$$

$$= 0$$

<u>Example</u> $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

$\exp_N$ is defined on all of $T_NS^2 = \{(X,Y,1) \mid X,Y \in \mathbb{R}\}$



$\exp_N$ is a diffeo on $B_\pi = \{(X,Y,1) \mid X^2+Y^2<\pi^2\}$