LECTURE - 9

- RIEMANNIAN DISTANCE FUNCTION: (M, g) Riemannian manifold.

Def^n: Let $\gamma : [a,b] \longrightarrow M$ be a cont. curve. We say.

1) γ is <u>regular</u> if γ is <u>smooth</u> and $\dot{\gamma}(t) \neq 0$ $\forall$ $t \in (a,b)$.

2) γ is <u>admissible</u> if it is piecewise regular i.e. $\exists$ a finite sub-division $a = a_0 < a_1 \cdots < a_n = b$ s.t. $\gamma|_{[a_{k-1}, a_k]}$ is a regular curve.

For convenience, we also allow constant curves $\gamma(t) = p$ $\forall t$ to be admissible.

We denote

$$\left.\begin{aligned} \dot{\gamma}(a_k^-) &:= \lim_{t \to a_k^-} \dot{\gamma}(t) \\ \dot{\gamma}(a_k^+) &:= \lim_{t \to a_k^+} \dot{\gamma}(t) \end{aligned}\right\} \quad \boxed{\Delta_k \dot{\gamma} = \dot{\gamma}(a_k+) - \dot{\gamma}(a_k-)}$$

$\Omega(M; p, q)$.

$\quad \parallel$

$\Omega(p,q) = \{\gamma : [a,b] \longrightarrow M \mid \gamma \text{ is admissible}, \gamma(a) = p, \gamma(b) = q\}$

Def^n (Length & Energy) Let γ be an admissible curve as above.

1) The <u>length</u> of γ is defined to be

$$L(\gamma) := \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} |\dot{\gamma}(t)| \, dt$$

2) The <u>energy</u> of γ is defined to be

$$\mathcal{E}(\gamma) := \frac{1}{2} \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} |\dot{\gamma}(t)|^2 \, dt.$$

Rk: If $\gamma \in \Omega(p,q)$, & $\gamma : [a,b] \to M$, then

$$L(\gamma)^2 \leq (b-a)\,\mathcal{E}(\gamma).$$

w/ equality $\iff$ $|\dot{\gamma}(t)| = 1 \ \forall t$.

Def^n: Let $\gamma : [a,b] \to M \in \Omega(p,q)$. A re-parametrization is another admissible curve $\tilde{\gamma} : [c,d] \to M$ s.t. $\tilde{\gamma} = \gamma \circ \varphi$ where $\varphi : [c,d] \to [a,b]$ is a homeomorphism & $\exists$ a sub-division $c = c_0 < c_1 \cdots < c_n = d$ s.t. $\varphi|_{[c_{k-1}, c_k]}$ is a diffeomorphism between $[c_{k-1}, c_k]$ & $[a_{k-1}, a_k]$. We say that φ is orientation preserving if $\varphi'(s) > 0 \ \forall \ s \neq c_k$, $k = 0, \ldots, n$ or orientation reversing otherwise.

Lemma 9.1 1) If $\tilde{\gamma}$ is a reparametrization of $\gamma \in \Omega(M, p, q)$. Then $L(\tilde{\gamma}) = L(\gamma)$.

2) Given any $\gamma \in \Omega(p,q)$, $\exists$ reparametrization $\tilde{\gamma}$ s.t $|\tilde{\gamma}'(s)| \equiv 1$.

Pf^n: 1) is simply change of variable. For 2), let

$$s(t) = \int_a^t |\dot{\gamma}(u)|\, du.$$

Then s is increasing, cont. function on $[a,b]$ & hence a homeo from $[a,b] \to [0, L]$ where $L = L(\gamma)$. Let φ be the inverse.

Fundamental Th^m $\Rightarrow$ s is smooth on $[a,b] \setminus \{a_0, \dots, a$
& cont on $[a,b]$. Moreover, $s'(t) = |\dot{r}(t)| \neq 0$.

Inverse function theorem $\Rightarrow$ φ is smooth on $s(a_{k-1}, a_k)$ $\forall$ k & $\varphi'(s) = 1/|\dot{r}(\varphi(s))|$. So $\tilde{r} = r \circ \varphi$ is a reparametrization s.t

$$|\tilde{r}'(s)| = |r'(\varphi(s))| \cdot |\varphi'(s)| = 1.$$

<u>Def^n</u>: The <u>Riemannian</u> <u>distance</u> <u>function</u> is defined to be.

$$d_g(p,q) := \inf_{r \in \Omega(p,q)} \mathcal{L}(r).$$

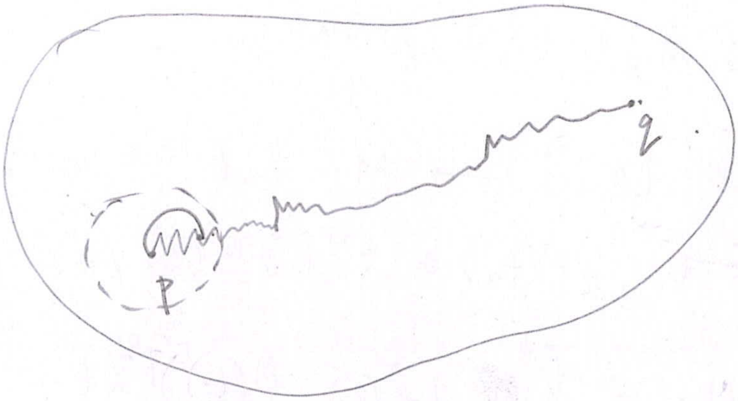
(We write $d_g = d$ if there is no confusion).

<u>Prop 9.2</u>: 1). d is well defined i.e $\Omega(p,q)$ is non-empty

2). (M, d) is a metric sp.

3). The induced topology is the same as the manifold topology.

<u>Pf</u> 1) M connected $\Rightarrow$ $\exists$ a <u>cont</u> path r s.t $r(0) = p$, $r(1) = q$.



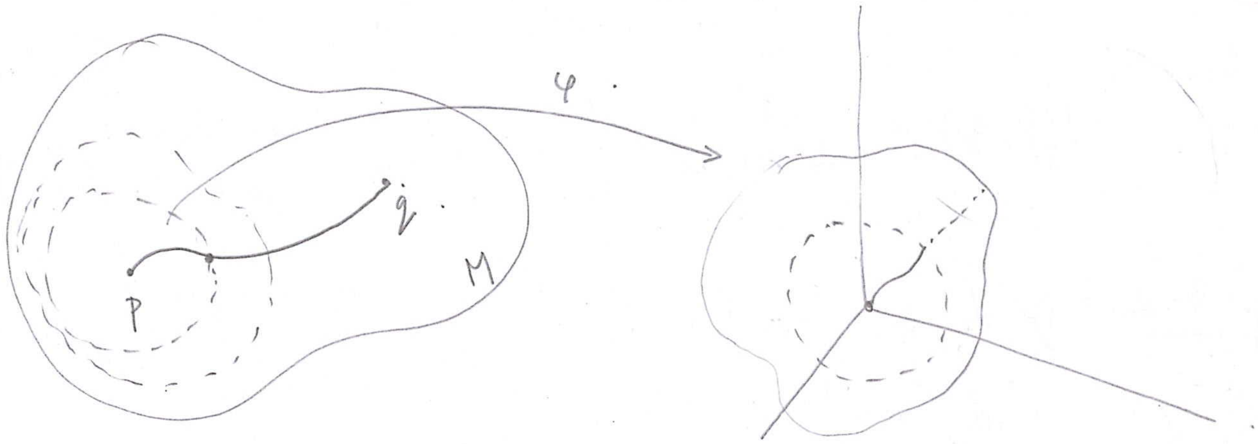
Compactness of $r[0,1]$ $\Rightarrow$ $\exists$ sub-division $a_0 = 0 < \dots$
$\dots < a_n = 1$ s.t $r[a_k, a_{k+1}]$ contained in one.

coordinate nbd. (U, φ). Connect $\varphi(\gamma(a_k))$, $\varphi(\gamma(a_{k+1}))$ by a st. line path & pull back to the mfld. Replace $\gamma(t)$, $a_k < t < a_{k+1}$ by this new path.

2). Clearly $d(p,q) = d(q,p) \geq 0$ & $d(p,p) = 0$. Also Δ- ineq. is easy to prove

<u>Claim</u> $d(p,q) > 0$ if $p \neq q$.

<u>Pf</u>:



Let (U, φ) be a coordinate chart centered at $p \in M$ s.t $\varphi(p) = 0$, $\varphi(U) = B(0,1) \subset \mathbb{R}^n$, $q \notin U$. (Hausdorff prop.) Let $h = (\varphi^{-1})^* g$ induced metric on $B(0,1)$. $B(0,1/2) \subset\subset B(0,1) \implies \exists\ \lambda$ & $\mu > 0$ s.t $\forall\ u \in \mathbb{R}^n$, $\forall\ x \in B(0,1/2)$

$$\lambda \|u\|^2 \leq h_x(u,u) \leq \mu \|u\|^2$$

Let $\gamma \in \Omega(p,q)$, $\gamma: [a,b] \longrightarrow M$. Let $t_0 \in [a,b]$ be the <u>first</u> <u>pt</u> s.t $\gamma(t_0) \in f^{-1}(\partial B(0,1/2))$. Then if $c = \varphi \circ \gamma$,

$$L(\gamma) \geq \int_0^{t_0} |\dot{\gamma}(t)|\, dt = \int_0^{t_0} [g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))]^{1/2} dt$$

$$= \int_0^{t_0} [h_{c(t)}(c'(t), c'(t))]^{1/2} dt$$

$$\geq \sqrt{\lambda} \int_0^{t_0} |c'(t)| dt \geq \sqrt{\lambda} \| c(t_0) - c(0) \| = \sqrt{\lambda}/2 > 0 .$$

3) <u>Claim</u> Let $p, U, \varphi, \lambda, \mu$ as above. Then $\forall$ $x \in f^{-1}(B(0, 1/2))$,

$$\sqrt{\lambda} \| \varphi(x) \| \leq d(p, x) \leq \sqrt{\mu} \| \varphi(x) \| . \quad (*)$$

Assuming this, let $V \subset_{open} M$ & $p \in V$.
One can choose $U \subset V$. If $x \in M$ s.t $d(p, x) < \sqrt{\lambda}/2$
Then $\exists$ a curve $\gamma : [a, b] \to M$ s.t. $L(\gamma) < \sqrt{\lambda}/2$.
Let t_0 first time s.t $x_0 = \gamma(t_0) \in f^{-1}(\partial B(0, 1/2))$.
By $(*)$ $d(p, x_0) \geq \sqrt{\lambda} \| \varphi(x_0) \| = \sqrt{\lambda}/2$.
In particular

$$\sqrt{\lambda}/2 > L(\gamma) \geq \int_a^{t_0} |\dot{\gamma}(t)| \geq d(p, x_0) \geq \sqrt{\lambda}/2 .$$

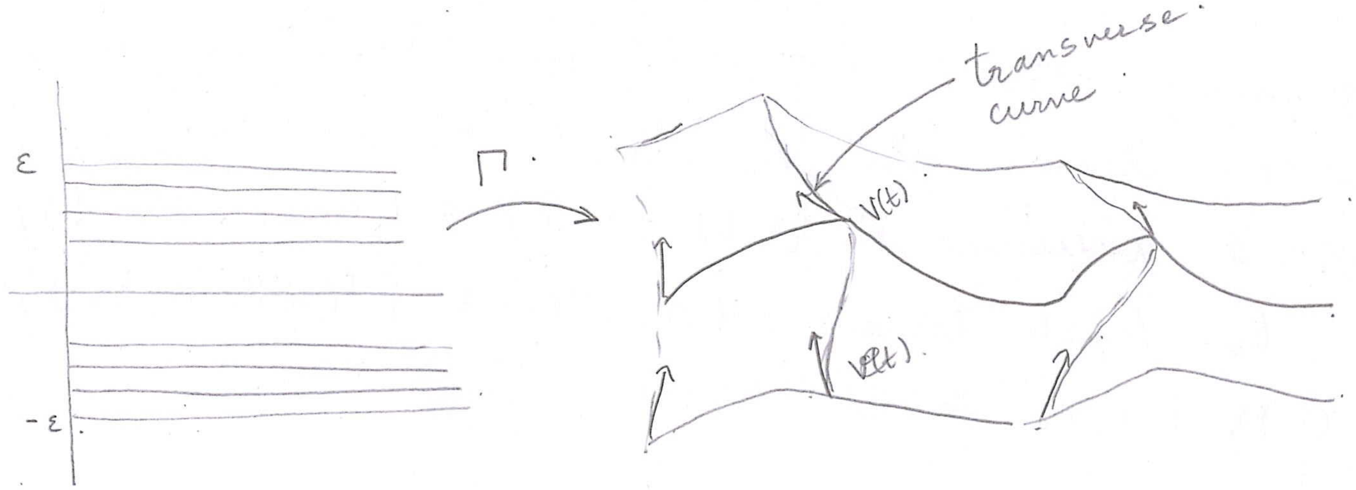
Contradiction. Similarly if $B_d(p, \varepsilon)$ is a ball in the metric, one can show $\forall q \in B_d(p, \varepsilon)$, $\exists$ a $q \in U \subset B_d(p, \varepsilon)$ s.t $U \subset_{open} M$.
$(*)$ is proved by estimates as in the proof of Claim above.

• FIRST VARIATION FORMULA.

Def^n: An <u>admissible family</u> of <u>curves</u> is a cont. map $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M$ w/.

(1) $\exists$ a sub-division $a = a_0 < a_1 \cdots < a_n = b$ s.t. Γ is smooth on $(-\varepsilon, \varepsilon) \times [a_{k-1}, a_k]$, $k = 1, \cdots, n$.

(2). $\gamma_s(t) := \Gamma(s,t)$ is an admissible curve $\forall\, s \in (-\varepsilon,$



Def^n: Let $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M$ be an admissible family,

1). An <u>admissible vector field</u> along Γ is a cont. map $V : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow TM$ s.t $V(s,t) \in T_{\Gamma(s,t)}M$ & a sub-division $a = \tilde{a}_0 < \tilde{a}_1 \cdots < \tilde{a}_n = b$ s.t. $V\big|_{(-\varepsilon,\varepsilon) \times [\tilde{a}_{k-1}, \tilde{a}_k]}$ is smooth.

2). The <u>transverse</u> curves are defined by $\Gamma^{(t)}(s) = \Gamma(s,t)$ & we define

$$\partial_t \Gamma(s,t) := \frac{d}{dt} \gamma_s(t), \quad \partial_s \Gamma = \frac{d}{ds} \Gamma^{(t)}(s).$$

Rk: $\Gamma^{(t)}(s)$ is smooth $\forall$ $s \in (-\varepsilon, \varepsilon)$.

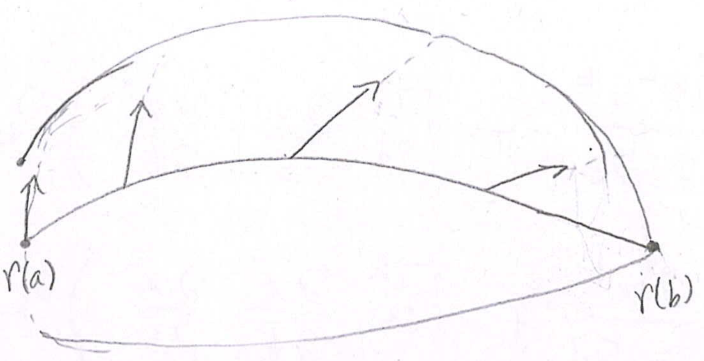
Def^n: 1) If $\gamma : [a,b] \to M$ is an admissible curve, a variation of γ is an admissible family Γ s.t $\gamma_0(t) := \Gamma(0,t) = \gamma(t)$. It is called proper if $\gamma_s(a) = \gamma(a)$ & $\gamma_s(b) = \gamma(b)$ $\forall$ $s \in (-\varepsilon, \varepsilon)$.

2) If Γ is a variation of γ, we call $V(t) := \partial_s \Gamma(0,t)$ the variation field of Γ. A vector field $V(t)$ along γ is called proper if $V(a) = V(b) = 0$.

Rk: If Γ is proper, then it's variation field is proper.

Lemma 9.3: If γ is admissible & V is a v.f along γ, then $\exists$ a variation Γ s.t $V(t) = \partial_s \Gamma(0,t)$. If V is proper, we can take Γ to be proper.

Pf: Define $\Gamma(s,t) = \exp_{\gamma(t)}(s \cdot V(t))$, $\gamma : [a,b] \to M$



Compactness of $[a,b] \Rightarrow \exists$ ε s.t $\Gamma(s,t) : (-\varepsilon, \varepsilon) \to [a,$

Clearly Γ is smooth on $(-\varepsilon, \varepsilon) \times [a_{k-1}, a_k]$ on each sub-interval $[a_{k-1}, a_k]$ on which V is smooth.

Moreover, if $V(a) = V(b) = 0$, then $\Gamma(s,a) \equiv \gamma(a)$
$\Gamma(s,b) \equiv \gamma(b)$.

<u>Th^m 9.4</u> (First variation formula). Let $\gamma : [a,b] \to M$ be <u>any</u> ~~unit speed~~ admissible curve, Γ a variation of γ, and V it's variation field. Then

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}(\gamma_s) = -\int_a^b \langle V, D_t \dot{\gamma} \rangle + \langle V(b), \dot{\gamma}(b-) \rangle - \langle V(a), \dot{\gamma}(a+) \rangle - \sum_{k=1}^{n-1} \langle V(a_k), \Delta_k \dot{\gamma} \rangle$$

For the proof we need the foll lemma.

<u>Lemma 9.5</u> (Symmetry lemma). Let Γ be an admissible family. Then on $(-\varepsilon, \varepsilon) \times [a_{k-1}, a_k]$,

$$D_s \partial_t \Gamma = D_t \partial_s \Gamma.$$

<u>Pf</u>: Compute in local coordinates on $U, (x^i)$. Let $\Gamma(s,t)|_U = (x^1(s,t), \dots, x^n(s,t))$. Then

$$\partial_t \Gamma = \frac{\partial x^k}{\partial t} \partial_k, \quad \partial_s \Gamma = \frac{\partial x^k}{\partial s} \partial_k.$$

So

$$D_s \partial_t \Gamma = \left(\frac{\partial^2 x^k}{\partial s \partial t} + \Gamma^k_{ij} \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial t} \right) \partial_k.$$

$$D_t \partial_s \Gamma = \left(\frac{\partial^2 x^k}{\partial s \partial t} + \Gamma^k_{ji} \frac{\partial x^j}{\partial t} \frac{\partial x^i}{\partial s} \right) \partial_k.$$

Done, since $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Pf of Th^m 9.4 : Recall.

$$\mathcal{E}(\gamma_s) := \frac{1}{2}\sum_{k=1}^{n}\int_{a_{k-1}}^{a_k}\langle \dot{\gamma}_s, \dot{\gamma}_s\rangle\, dt \qquad \left(\dot{\gamma}_s = \frac{d}{dt}\gamma_s\right)$$

So,

$$\frac{d}{ds}\Big|_{s=0}\mathcal{E}(\gamma_s) = \sum_{k=1}^{n}\int_{a_{k-1}}^{a_k}\langle D_s\partial_t\Gamma(0,t), \dot{\gamma}\rangle\, dt.$$

$$= \sum_{k=1}^{n}\int_{a_{k-1}}^{a_k}\langle D_t\partial_s\Gamma(0,t), \dot{\gamma}\rangle\, dt \quad \text{(Symmetry lemma)}.$$

$$= \sum_{k=1}^{n}\int_{a_{k-1}}^{a_k}\frac{d}{dt}\langle V(t), \dot{\gamma}(t)\rangle\, dt - \sum_{k=1}^{n}\int_{a_{k-1}}^{a_k}\langle V(t), D_t\dot{\gamma}\rangle\, dt$$

$$= -\int_a^b\langle V(t), D_t\dot{\gamma}\rangle\, dt + \sum_{k=1}^{n}\Big[\langle V(a_k), \dot{\gamma}(a_k-)\rangle - \langle V(a_{k-1}), \dot{\gamma}(a_{k-1}+)\rangle\Big]$$

$$= -\int_a^b\langle V(t), D_t\dot{\gamma}\rangle\, dt + V(b)\dot{\gamma}(b-) - V(a)\dot{\gamma}(a+) - \sum_{k=1}^{n-1}\langle V(a_k), \Delta_k\dot{\gamma}\rangle.$$