MA333: Assignment-1

(due August 19, 2019)

Note. Submit solutions to 1(e), 2, 4(d), 5(e) and 6(e). Please submit solutions if you plan to credit the course. **If there is a possibility that you might credit the course, please do submit the solutions, since there will no late submission allowed**.

1. Let vector bundles E and F be vector bundles of rank r and s respectively on M, and let $\nabla^{(E)}$ and $\nabla^{(F)}$ be linear connections on them, thought of as maps from $\Gamma(E)$ to $\Gamma(T^*M \otimes E)$ and $\Gamma(F)$ to $\Gamma(T^*M \otimes F)$ respectively.
   (a) Prove that $E \oplus F := \coprod_{p \in M} E_p \oplus F_p$ can be given the structure of a vector bundle of rank $r+s$ such that the natural inclusion maps $i_E : E \to E \oplus F$ and $i_F : F \to E \oplus F$ are smooth, and with the following universal property: For any vector bundle G, and vector bundle maps $f_E : E \to G$ and $f_F : F \to G$, there exists a unique map $f : E \oplus F \to G$ such that $f \circ i_E = f_E$ and $f \circ i_F = f_F$.
   (b) Prove that $E \otimes F := \coprod_{p \in M} E_p \otimes F_p$ can be give the structure of a vector bundle with an associated smooth bilinear map $\varphi : E \oplus F \to E \otimes F$ (here bi-linearity is fibre-wise) such that the following universal property holds: For every vector bundle G, and every smooth bi-linear map $B \to E \oplus F \to G$, there exists a unique smooth linear map $\tilde{B} : E \otimes F \to G$ such that $B = \tilde{B} \circ \varphi$.
   (c) Show that $\nabla^{E \otimes F} := \nabla^{(E)} \otimes \mathbf{1}_F + \mathbf{1}_E \otimes \nabla^{(F)}$ defines a linear connection on $E \otimes F$.
   (d) Show that $E^* := \coprod_{p \in M} E_p^*$ and $End(E) := \coprod_{p \in M} End(E_p)$ can be given structures of vector bundles such that $End(E)$ is isomorphic to $E \otimes E^*$.
   (e) Show that $\nabla^{(E)}$ naturally induces a linear connections $\nabla^{(E^*)}$ and $\nabla^{(End(E))}$ on E^* and $End(E)$ in a similar way to how we defined connections on tensor bundles. Write down the formulae for the Christoffel symbols in terms of the corresponding symbols for $\nabla^{(E)}$.

2. Let ∇ be a linear connection on M with Christofell symbols A_{ij}^k, and we continue to denote it's extension to tensor bundles by ∇. Prove that

$$\nabla_i T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} = \partial_i T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} + \sum_{p=1}^{r} A^{i_p}_{ij} T^{i_1 \cdots j \cdots i_r}{}_{j_1 \cdots j_s} - \sum_{q=1}^{s} A^k_{ij_q} T^{i_1 \cdots i_r}_{j_1 \cdots k \cdots j_s}.$$

   Here the j in the second term and the k in the third term replace i_p and j_q respectively in the components of T.

3. (a) Let $\Omega \subset \mathbb{R}^2$ be open and let $f : \Omega \to \mathbb{R}$ be a smooth function. Let S be the surface defined by $z = f(x, y)$. Calculate the induced Riemannian metric g on S. Prove that the induced volume form is given by

$$dV_g = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy.$$

   (b) More generally, let S be a smooth parametric surface with parametrization $\vec{r} : \Omega \to \mathbb{R}^3$ for some $\Omega \subset \mathbb{R}^2$. Calculate the induced metric on S, and prove that the corresponding volume form (or the surface area element) is given by

$$dS = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dudv.$$

4. (**Killing fields.**) Let (M, g) be a compact Riemannian manifold. A vector field X is said to be *Killing* if $\mathcal{L}_X g = 0$.
   (a) Calculate $\mathcal{L}_X g$ in local coordinates.
   (b) Let φ_t be the flow of X. That is, $\varphi_0(x) = x$ for all $x \in M$, and

$$\frac{d\varphi_t(x)}{dt} = X(\varphi_t(x)).$$

   Prove that X is Killing if and only if φ_t is an isometry for all t.

(c) If X and Y are Killing fields, prove that $[X, Y]$ is also a Killing field. These two exercises demonstrate that the set of all Killing fields is isomorphic to the Lie-algebra of the isometry group.

(d) Show that on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the vector fields

$$x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \ (1 \leq i < j \leq n+1),$$

are Killing fields. Can you recognize the isometries generated by them? **Note.** You first have to show that the vector fields are indeed tangential to the sphere.

5. A Lie group is a finite dimensional manifold G such that the group operations (namely products and taking inverses) is smooth. The Lie algebra $\mathfrak{g}$ is defined to the tangent space of G at the identity e. A Riemannian metric is said to be *left (resp. right) invariant* if for any $x \in G$, $L_x^* g = g$ (resp. $R_x^* g = g$), where $L_x(y) = x \cdot y$ (resp. $R_x(y) = yx$) denotes the multiplication on the left (resp. right). It is said to be bi-invariant if it is invariant on the left and the right. Conjugation $C_x(y) = xyx^{-1}$ by a group element defines an *inner automorphism.* The derivatives $\mathrm{Ad}_x := dC_x : \mathfrak{g} \to \mathfrak{g}$ of such automorphisms define a representation $\mathrm{Ad} : G \times g \to \mathfrak{g}$ called the *adjoint representation* of G.

(a) Prove that $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are all Lie groups. Can you identify their Lie algebras, and dimensions?

(b) Show that a metric g on G is left invariant if and only if for any left invariant frame $\{X_i\}$, the co-efficients $g_{ij} := g(X_i, X_j)$ are constant functions.

(c) Show that the restriction $g \to g|_{\mathfrak{g}}$ gives a bijection between left invariant metrics g on G and left invariant inner products on $\mathfrak{g}$. Moreover, an inner product on $\mathfrak{g}$ induces a bi-invariant metric on G if and only if the inner product is invariant under the adjoint representation (that is Ad_x is an isometry of $\mathfrak{g}$ for all $x \in G$).

(d) If g is a left invariant metric on G, prove that the corresponding volume form dV_g is bi-invariant.

(e) Use the above two parts to conclude that on any compact connected Lie group, there always exists a bi-invariant Riemannian metric. **Hint.** If you can get a left invariant metric on G, then using an averaging trick and the part above, one can construct a bi-invariant metric on $\mathfrak{g}$.

6. The aim of this question is to characterise all flat two dimensional tori upto isometries. Recall that a two dimensional torus (as a smooth manifold) is defined to be $\mathbb{T} := \mathbb{S}^1 \times \mathbb{S}^1$.

(a) If $\{\tau_1, \tau_2\}$ is any basis of $\mathbb{R}^2$, and $\Lambda = \{n\tau_1 + m\tau_2 \mid (n, m) \in \mathbb{Z}^2\}$ is the corresponding lattice, then prove that $\mathbb{T}$ is diffeomorphic to $\mathbb{R}^2/\Lambda$ by producing an explicit diffeomorphism.

(b) For each Λ, the Euclidean metric (being translation invariant) descends to a Riemannian metric g_Λ on $\mathbb{T}$. If $(\theta_1, \theta_2) \in (-\pi, \pi)^2$ are canonical coordinates on $\mathbb{T} \setminus (\{(-1, 0)\} \times \mathbb{S}^1 \cup \mathbb{S}^1 \times \{(-1, 0)\})$, then prove that

$$g_\Lambda = \sum_{i,j=1}^{2} \langle \tau_i, \tau_j \rangle d\theta^i \otimes d\theta^j,$$

where $\langle \cdot \rangle$ is the Euclidean inner product.

(c) Prove that $(\mathbb{T}, g_\Lambda)$ is isometric to $(\mathbb{T}, g_{\Lambda'})$ if and only if there exists an isometry of $\mathbb{R}^2$ that sends Λ to Λ'. We say $\mathbb{T}$ is a square (resp. rectangular or hexagonal) flat tori if it in equipped with the metric g_Λ, where $\tau_1 = (1, 0)$ and $\tau_2 = (0, 1)$ (resp. $\tau_2 = (0, a)$, $a > 1$ or $\tau_2 = (1/2, \sqrt{3}/2)$).

(d) We say two metrics g_1 and g_2 on a manifold are *homothetic* if there exists a $\lambda > 0$ such that $g_2 = \lambda g_1$. Show that the equivalence classes of homothetic metrics of the form g_Λ on $\mathbb{T}$ are in bijective correspondence to

$$\mathcal{M} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1, \ 0 \leq x \leq 1/2, \ y > 0\}.$$

Hint. By applying a rotation and dilation, one can assume that $\tau_1 = (1, 0)$.

(e) Show that $\Phi : \mathbb{T} \to \mathbb{S}^3$ defined by $\Phi(\theta, \varphi) = (e^{i\theta}, e^{i\varphi})$ defines an embedding of $\mathbb{T}$ in $\mathbb{S}^3$. The resulting submanifold is called the *Clifford torus.* Show that the Riemannian metric on $\mathbb{T}$ induced from the usual round metric on $\mathbb{S}^3$ makes $\mathbb{T}$ into a square torus. **Note.** This shows that there is an isometric embedding of the square torus into $\mathbb{R}^4$. In a later exercise, we'll show that there does not exist any C^2 isometric embedding of a flat torus in $\mathbb{R}^3$, though most remarkably, by a theorem of Nash-Kuiper, there does exist a C^1 isometric embedding. There are beautiful animations of this on youtube.