MA333: Assignment-2

(due 09/09/19)

Note. Please submit solutions to problems 3(b), (e), (f), 4(c), 5(e), 6(c), (d).

1. (Hyperbolic spaces comparison) Recall that in lecture, we defined the Riemannian metrics $g_{\mathbb{B}^n_R}$ and $g_{\mathbb{U}^n_R}$ on the open unit ball $\mathbb{B}^n_R \subset \mathbb{R}^n$ and the upper half space $\mathbb{U}^n_R \subset \mathbb{R}^{n+1}$ respectively.
   (a) Let $\mathbb{R}^{n+1}$ be equipped with the quadratic form $Q(x) = -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2$, and let $\mathbb{H}^n_R := \{x = (x^0, \cdots, x^n) \mid Q(x) = -R^2,\ x^0 > 0\}$. Show that the bi-linear form
   $$-d(x^0)^2 + d(x^1)^2 + \cdots + d(x^n)^2$$
   induces a Riemannian metric on $\mathbb{H}^n_R$. We denote this metric by $g_{\mathbb{H}^n_R}$.
   (b) Prove that $(\mathbb{B}^n_R, g_{\mathbb{B}^n_R})$ and $(\mathbb{H}^n_R, g_{\mathbb{H}^n_R})$ are isometric. **Hint.** For $n = 2$, this is simply the Cayley transform from complex analysis. Writing down it's real and imaginary parts, should suggest a higher dimensional generalization. For simplicity, first you can assume $R = 1$, and then do the general case by a scaling argument.
   (c) Find an isometry from $(\mathbb{H}^n_R, g_{\mathbb{H}^n_R})$ to $(\mathbb{B}^n_R, g_{\mathbb{B}^n_R})$. **Hint.** Take the stereographic projection from $\mathbb{H}^n_R$ to $(-R, 0, \cdots, 0)$.

2. (a) Let C_α be the cone of angle 2α with vertex at the origin. More precisely, C_α is the cone $x^2 + y^2 = z^2 \tan^2 \alpha,\ z > 0$. Endow $C_\alpha \setminus \{0\}$ with the Riemannian metric g_α which is the restriction of the Euclidean metric. Then prove that (C_α, g_α) is isometric to a sector in $\mathbb{R}^2$ of angle $2\pi \sin \alpha$ by exhibiting an explicit map.
   (b) Let $\gamma : [0, 1] \to C_\alpha$ be any circle $z = c$ on the cone, and let $X(t)$ be any parallel unit vector field along γ. Calculate the angle between $X(0)$ and $X(1)$.
   (c) Calculate the Christoffel symbols of the Levi-Civita connection of the round metric on $\mathbb{S}^2$ in spherical coordinates.
   (d) Let $\gamma : [0, 1] \to \mathbb{S}^2$ be the curve $\varphi = \varphi_0$ and $\theta = 2\pi t$ (in spherical coordinates), and let v be any tangent vector to $\mathbb{S}^2$ at $\gamma(0)$. Calculate the angle between v and $P_{\gamma,0,1}(v)$ in terms of φ_0, where P_γ is the parallel transport with respect to the Levi-Civita connection of the round metric on $\mathbb{S}^2$.

3. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold, and let $M \subset \tilde{M}$ be a smooth hypersurface (ie. a submanifold of codimension one) with the induced Riemannian metric g_M. Denote the respective Levi-Civita connections by $\tilde{\nabla}$ and ∇. Set $N_pM := (T_pM)^\perp$, the orthogonal complement of T_pM with respect to $\tilde{g}$.
   (a) Show that $NM := \coprod_{p \in M} N_pM$ can be given the structure of a rank one vector bundle such that $T\tilde{M}|_M \cong TM \oplus NM$ as vector bundles. Moreover, prove that the orthogonal projections $\pi^\top : T\tilde{M}\Big|_M \to TM$ and $\pi^\perp : T\tilde{M}\Big|_M \to NM$ are smooth. We denote the set of smooth sections of NM by $\mathcal{N}(M)$.
   (b) The *second fundamental form* $II(X, Y) : \mathcal{T}^1(M) \times \mathcal{T}(M) \to \mathcal{N}(M)$ is defined by
   $$II(X, Y) := (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^\perp,$$

where $\tilde{X}$ and $\tilde{Y}$ are smooth extensions of X and Y to $\mathcal{T}^1(\tilde{M})$. Prove that $II(X,Y)$ is independent of the extensions and defines a smooth, symmetric, $C^\infty(M)$ bilinear map. Moreover,

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \nabla_X Y + II(X,Y).$$

(c) (**Weingarten equation.**) Suppose $X, Y \in \mathcal{T}^1(M)$ and $N \in \mathcal{N}(M)$, and let $\tilde{X}, \tilde{Y}$ and $\tilde{N}$ be arbitrary extensions to $\tilde{M}$, the prove that

$$\tilde{g}\left(\tilde{\nabla}_{\tilde{X}}\tilde{N}, \tilde{Y}\right) = -g(N, II(X,Y)).$$

(d) For the problems to follow, we specialise to the case when $\tilde{M} = \mathbb{R}^{n+1}$ with the Euclidean metric, and assume that M is orientable.We denote the inner product on $\mathbb{R}^{n+1}$ by $\langle \cdot, \cdot \rangle$. Prove that there exists a nowhere vanishing smooth section $\vec{N} \in \mathcal{N}(M)$. In particular, NM is a trivial line bundle.

(e) We normalize so that $\langle \vec{N}, \vec{N} \rangle = 1$. The *scalar second fundamental form* is defined by $h(X,Y) = \langle II(X,Y), \vec{N} \rangle$. If $\gamma : (-\varepsilon, \varepsilon) \to M$ is any smooth curve, show that

$$\ddot{\gamma}(t) = \mathcal{D}_t \dot{\gamma}(t) + h(\dot{\gamma}, \dot{\gamma})(t)\vec{N}(\gamma(t)),$$

and hence γ is a geodesic if and only if $\ddot{\gamma}$ is everywhere normal to M.

(f) Let $\Omega \to \mathbb{R}^{n+1}$ be an open set, $F : \Omega \to \mathbb{R}$ be a smooth submersion (ie. $\nabla F \neq 0$ on all of Ω), and $M = F^{-1}(0)$ be non-empty. Show that the scalar second fundamental form with respect to the unit normal $\vec{N} = \nabla F / |\nabla F|$ is given by

$$h(V,W) = -\frac{V^i W^j \partial_i \partial_j F}{|\nabla F|^2}.$$

4. Let $c : [0,1] \to \mathbb{R}^2$ be a plane curve parametrised by arc length such that c is injective, and set $c(t) = (r(u), z(u))$ with $r > 0$. The surface S obtained by rotating the image of c can be parametrized by

$$(r(u)\cos\theta, r(u)\sin\theta, z(u)),$$

where θ is the rotation angle.

(a) Show that the metric induced on S from $\mathbb{R}^3$ is given by $g = du^2 + r^2(u)d\theta$.

(b) Calculate the Chritoffel symbols with respect to the coordinates (u, θ).

(c) Show that the geodesics are

1. Meridians obtained by intersecting the surface by planes containing the axis of revolution.
2. Parallels ($u = \text{const}$) for which $r'(u) = 0$.
3. The curves $(u(t), \theta(t))$ which when parametrized by arc-length satisfy

$$\left(\frac{du}{dt}\right)^2 + r^2(u(t))\left(\frac{d\theta}{dt}\right)^2 = 0, \text{ and, } r^2(u(t))\frac{d\theta}{dt} = C,$$

where C is some constant associated to the geodesic.

5. (**Divergence of a vector field.**) Let (M, g) be a Riemannian manifold. The *divergence* of a vector field X is defined to be $\text{div}(X) := \text{tr}(\nabla X) = \nabla_i X^i$. The *Hessian* is defined by $\text{Hess}(f)(X,Y) := \nabla_X \nabla_Y f$. The *Laplacian* of a function f is defined to be $\Delta f = \text{tr}(\text{Hess} f) = \nabla_i \nabla^i f$.

(a) If dV_g is the volume form of g, prove that $L_X dV_g = (\text{div}X)dV_g$, and hence prove that in local coordinates,

$$\text{div}X = \frac{1}{\det g}\partial_i(\sqrt{\det g}X^i).$$

(b) Calculate the divergence of the vector field

(c) Show that the Hessian is a section of S^2T^*M, and that $\mathrm{Hess}(f)(X,Y) = g(\nabla_X \nabla f, Y)$.

(d) Prove that for any function u, and any vector field X, we have,

$$\mathrm{div}(uX) = \langle \nabla u, X\rangle + u\mathrm{div}(X).$$

(e) Prove that on $\mathbb{R}^n \setminus \{0\}$,

$$\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}},$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian of the round metric on $\mathbb{S}^{n-1}$. Use this to prove that the restriction of the coordinate functions x, y and z to $\mathbb{S}^2$ define eigenfunctions of $\Delta_{\mathbb{S}^2}$. What are the corresponding eigenvalues?

6. (**Analysis on manifolds with boundary.**) Let M be a manifold with boundary ∂M. Note that ∂M is a submanifold of M., and for each $p \in \partial M$, there exists a coordinate chart (U, φ) such that $\varphi(U) = \mathbb{B}^n_+ := \{(x^1, \cdots, x^n) \mid \sum |x^i|^2 < 1,\ x^n \geq 0\}$ with $\varphi(p) = 0$. We have the tangent spaces $T_p\partial M = \mathrm{sp}(\partial_1, \cdots, \partial_{n-1}) \subset T_pM := \mathrm{sp}(\partial_1, \cdots, \partial_n)$. A metric g on M (defined in the same way as in the case of manifolds without boundary) restricts to a metric $\tilde{g}$ on ∂M. We say that $\vec{N} \in \mathcal{N}(\partial M)$ is *inward pointing* at $p \in \partial M$ if there exists a curve $\gamma : [0, \varepsilon] \to M$ such that $\gamma(0) = p$ and $\gamma'(p) = \vec{N}$.

(a) Show that a normal field $\vec{N}$ is inward pointing if and only if in every chart (U, φ) as above, $\vec{N}$ has a strictly positive $\partial/\partial x^n$ component.

(b) Prove that there exists a unique *outward* pointing unit normal field $\vec{N}$ along ∂M (ie. $-\vec{N}$ is inward pointing along ∂M). Moreover, if M is oriented with the corresponding nowhere vanishing form Ω, then $i_{\vec{N}}\Omega$ induces an orientation on ∂M. We call this the positive orientation for ∂M.

(c) Prove that

$$dV_{\tilde{g}} = i_{\vec{N}} dV_g,$$

where we have used the positive orientation on ∂M to write down the volume form. **Sanity check.** If M is the unit disc in $\mathbb{R}^2$ and $\partial M = \mathbb{S}^1$ is the boundary circle, then $dV_g = r dr \wedge d\theta$. The outward unit normal if $\vec{N} = \partial/\partial r$, and $i_{\vec{N}} dV_g = d\theta$ is the standard length element on $\mathbb{S}^1$, and integration along the circle will be anti-clockwise.

(d) Let (M, g) be a compact oriented manifold with boundary $(\partial M, \tilde{g})$, and outward normal vector field $\vec{N}$. If X is any vector field on M, prove that

$$\int_M \mathrm{div} X \, dV_g = \int_{\partial M} \langle X, \vec{N}\rangle \, dV_{\tilde{g}}.$$

(e) Prove the following *Green's identities:*

$$\int_M u\Delta v \, dV + \int_M \nabla u \cdot \nabla v \, dV = \int_{\partial M} u \nabla v \cdot \vec{N} \, d\tilde{V} \qquad (1)$$

$$\int_M (u\Delta v - v\Delta u) \, dV_g = \int_{\partial M} (u\nabla v \cdot \vec{N} \ - v\nabla u \cdot \vec{N}) \, d\tilde{V}. \qquad (2)$$

(f) Hence prove that if u is *harmonic*, that is if $\Delta u = 0$, and M is *closed* (that is, without boundary), then u is a constant.