MA333: Assignment-3

(due 30/09/2019)

Note. Submit solutions to 3(b), (c), (f), (g), 4(b), 7

1. Give a direct proof (without using the exponential map) of the existence of coordinates $(x^1, \cdots, x^n)$ centered at any point p on a Riemannian manifold (M, g) such that the metric components satisfy $g_{ij}(p) = \delta_{ij}$ and $g_{ij;k}(p) = 0$ for all i, j, k. **Hint.** First use a linear change of coordinates to make the metric Euclidean at p, and then introduce a further quadratic change of coordinates to kill the linear term in the Taylor expansion of the metric.

2. Consider the unit sphere $\mathbb{S}^2$. The tangent space $T_N\mathbb{S}^2$ at the north pole can be identified with the plane $z = 1$ in $\mathbb{R}^3$. Write down an explicit formula for $\exp_N(\vec{v})$, for a vector $\vec{v} \in T_N\mathbb{S}^2$. What is the largest open set $\mathcal{V} \subset T_N\mathbb{S}^2$ for which the exponential map is a diffeomorphism?

3. Let G be a Lie group with Lie algebra $\mathfrak{g} \cong T_eG$, where e is identity of the group.
   (a) For every $\xi \in \mathfrak{g}$, prove that there exists a unique smooth homomorphism $\lambda_\xi : \mathbb{R} \to G$ such that $\lambda'_\xi(0) = \xi$. Moreover if X_ξ is the left invariant vector field on G induced by ξ, prove that $\lambda'_\xi(t) = X_\xi(\lambda(t))$ for all $t \in \mathbb{R}$.
   (b) One can define a Lie group exponential map $\mathrm{Exp} : \mathfrak{g} \to G$ by $\mathrm{Exp}(\xi) := \lambda_\xi(1)$. If $G \subset GL(n, \mathbb{R})$, prove that $\mathfrak{g} \subset M(n, \mathbb{R})$, and that
   $$\mathrm{Exp}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$
   Here $GL(n, \mathbb{R})$ is the multiplicative group of invertible $n \times n$ matrices and $M(n, \mathbb{R})$ is the algebra of all $n \times n$ matrices.
   (c) If G has a *binvariant* Riemannian metric (which is the case for instance if G is compact, by Assignment-1), then prove that Exp is equal to the Riemannian exponential map $\exp_e$ at the identity. **Hint.** First show that if ∇ is the Levi-Civita connection and X and Y are two left invariant vector fields, then $2\nabla_X Y = [X, Y]$. Hence prove that the geodesics are precisely the one-parameter subgroups $\lambda(t)$.
   (d) If $G \subset GL(n, \mathbb{R})$ is a Lie group, and X, Y are left invariant vector fields induced by matrices $X_e, Y_e \in \mathfrak{g} \subset M(n, \mathbb{R})$, prove that
   $$[X, Y]_e = X_0 Y_0 - Y_0 X_0.$$
   That is, the Lie algebra (or matrix) commutator agrees with the usual Lie bracket. For $\xi, \eta \in \mathfrak{g}$ giving rise to left invariant vector fields X_ξ, X_η we then simply put $[\xi, \eta] := [X_\xi, X_\eta]_e$. Also prove that the adjoint representation is given by conjugation. that is, for any $(g, A) \in G \times \mathfrak{g}$ (thought of as matrices),
   $$\mathrm{Ad}_g(A) = gAg^{-1}.$$
   (e) Prove that the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $SL(2, \mathbb{R})$ is spanned by basis
   $$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$
   Moreover, prove that $[H, X] = 2X$, $[X, Y] = H$ and $[H, Y] = -2Y$.

(f) Recall that a Lie group has a bi-invariant metric if and only if there is an inner product on $\mathfrak{g}$ which is invariant under the adjoint representation. Using this, prove that $SL(2,\mathbb{R})$ can have no bi-invariant Riemannian metric. **Hint.** Construct an matrix $g \in SL(2,\mathbb{R})$ such that $\mathrm{Ad}_g(X) = 4X$, where X is as above. Does that help?!

(g) Take the Frobenius inner product $\langle , \rangle$ on $\mathfrak{g}$ (so that $\langle A, B\rangle := \mathrm{tr}(A^T B)$) and extend this to a left-invariant Riemannian metric on $SL(2,\mathbb{R})$. Prove that the Riemannian exponential map at the identity is given by

$$\exp_e(A) = e^{A^T} e^{A-A^T},$$

and is not equal to the Lie group Exp defined above.

4. Let N be a *closed, embedded* submanifold of a Riemannian manifold (M, g). For any point $p \in M \setminus p$, we define the *distance of N from p* by

$$d(p, N) := \inf\{d(p, x) : x \in N\}.$$

(a) If (M, g) is complete, prove that N is also complete with the induced Riemannian metric.

(b) If $q \in N$ is a point such that $d(p, q) = d(p, N)$, and γ is any minimising geodesic from p to q, prove that q intersects N orthogonally. **Hint.** Use the first variation formula.

5. A curve $\gamma : [0, b) \to M$ $(0 < b \leq \infty)$ is said to *converge to infinity* if for every compact subset $K \subset M$, there is a time T such that $\gamma(t) \neq K$ for $t > T$. Prove that a Riemannian manifold (M, g) is complete if and only if every regular curve that converges to infinity has infinite length.

6. A Riemannian manifold (M, g) is said to be *homogenous* if there exists a Lie group G acting smoothly and transitively on M by isometries. It is called *isotropic at $p \in M$* if there exists a Lie group G acting smoothly on M by isometries such that the stabiliser subgroup $G_p \subset G$ (that is, the subgroup of elements that fix p) acts transitively (by pushforwards) on the set of unit tangent vectors in T_pM. Note that a homogenous Riemannian manifold that is isotropic at one point, is isotropic at all points.

(a) Show that a homogenous Riemannian manifold is necessarily complete.

(b) If (M, g) is a complete Riemannian manifold that is isotropic at each point. Show that M is homogenous.

7. A Riemannian submanifold $(M, g) \subset (\tilde{M}, \tilde{g})$ is called *totally geodesic* if for every $V \in TM$, the corresponding maximal $\tilde{g}$-geodesic $\tilde{\gamma}_V$ lies completely in M. Prove that the following are equivalent.

1. M is totally geodesic.
2. Every g-geodesic in M is also a $\tilde{g}$-geodesic in $\tilde{M}$.
3. The second fundamental form of M in $\tilde{M}$ (see Assignment-2 for the definition) vanishes identically.

8. A subset $\mathcal{U} \subset M$ is said to be *(geodesically) convex* if for every $p, q \in \mathcal{U}$, there is a unique (in M) minimising geodesic from p to q lying entirely in $\mathcal{U}$. The aim of this exercise is to show that for any $p \in M$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(p)$ is convex.

(a) Fix $p \in M$, and let $\mathcal{W}$ be a neighbourhood of p and $\varepsilon > 0$ such that any two points in $q_1, q_2 \in \mathcal{W}$ can be joined by a a minimal geodesic (given by $\gamma(t) = \exp_{q_1}(tv)$) of length smaller than ε. We proved the existence of such a neighbourhood in class. By shrinking ε further, we assume that $B(p, 2\varepsilon) \subset \mathcal{W}$, and define

$$\mathcal{W}_\varepsilon := \{(q, V, t) \in TM \times \mathbb{R} \mid q \in B(p, \varepsilon),\ V \in T_qM,\ |V| = 1,\ |t| < 2\varepsilon\},$$

and $f : \mathcal{W}_\varepsilon \to \mathbb{R}$ by $f(q, V, t) := d(\exp_q(tV), p)^2$. Prove that f is smooth. **Hint.** Use normal coordinates centred at p.

(b) Show that if ε is chosen small enough, then $\partial^2 f/\partial t^2 > 0$ on $\mathcal{W}_\varepsilon$. **Hint.** Compute $f(p, V, t)$ explicitly and then use continuity.

(c) If $q_1, q_2 \in B(p, \varepsilon)$ and γ is a minimizing geodesic from q_1 to q_2, prove that $d(\gamma(t), p)$ attains it's maximum at one of the end points.

(d) Show that $B(p, \varepsilon)$ is convex.