MA333: Assignment-4

Due 23/10/2019

Please submit solutions to problems 1, 3(d), 4, 6, 7 and 8.

1. Let (M, g) be a complete Riemannian manifold. Prove that for any $p \in M$, the cut-locus has the following characterisation:

$$Cut(p) = \{q \in M \mid \forall x \in M \setminus \{q\},\ d(p, q) + d(x, q) > d(p, x)\}.$$

2. Let (M, g) be a complete Riemannian manifold.

   (a) Prove that *cut-time* function $\tau_{cut} : SM \to M$ defined by

$$\tau_{cut}(\xi_p) = \begin{cases} t_0, & \text{if } \exp_p(t_0 \xi_p) \in Cut(p) \\ +\infty, & \text{otherwise,} \end{cases}$$

   is a continuous function.

   (b) Prove that $Cut(p)$ is a closed subset of M.

   (c) Let $\tilde{U}_p := \{t\xi_p \mid 0 \le t < \tau_{cut}(\xi_p),\ \xi_p \in S_pM\}$ and $\mathcal{U}_p := \exp_p(\tilde{U}_p)$ Prove that $M = \mathcal{U}_p \cup Cut(p)$ and $\mathcal{U}_p \cap Cut(p) = \phi$. **Note.** This was implicitly assumed and used in class, but I never provided a rigorous proof.

   (d) The boundary $\partial \tilde{U}_p$ is by definition the cut-locus $\tilde{Cut}(p)$ of p in the tangent space. Let

$$\partial^1 \tilde{U}_p := \{\xi \in \partial \tilde{U}_p \mid \exp_p(\xi) = \exp(\xi') \text{ for some } \xi' \in \partial \tilde{U}_p \setminus \{\xi\}\}.$$

   Prove that $\exp_p(\partial^1 \tilde{U}_p) \subset Cut(p)$ is dense, and that for any $q \in M \setminus \exp_p(\partial^1 \tilde{U}_p)$, there exists a unique geodesic connecting p and q.

   (e) Hence prove that if r_p is smooth in a neighbourhood of x, then $x \notin Cut(p)$. **Hint.** We have already shown in class that r_p is differentiable at x if and only if $x \in M \setminus \exp_p(\partial^1 \tilde{U}_p)$.

3. Let $E \to M$ be a smooth rank r vector bundle on M with a connection ∇. The operator d_∇ can be extended to an operator $d_\nabla : \Gamma(\Lambda^p T^*M \otimes E) \to \Gamma(\Lambda^{p+1} T^*M \otimes E)$ by setting

$$d_\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge d_\nabla s.$$

Define the *curvature* F_∇ of ∇ be setting $F_\nabla(s) = d^2_\nabla(s)$. We define the *curvature endomorphism* (also denoted by $F_\nabla : \mathcal{T}^1(M) \times \mathcal{T}^1(M) \times \Gamma(E) \to \Gamma(E)$ by

$$F_\nabla(X, Y)(s) := i_X i_Y d^2_\nabla s,$$

where the contraction is applied to the form part. That is, if ω is a local 2-form and s a local section, then $i_X i_Y(\omega \otimes s) := \omega(X, Y)s$. We also say that ∇ is *flat* if $F_\nabla \equiv 0$.

   (a) Prove that for any $f \in C^\infty(M)$ and any $s \in \Gamma(E)$, $F_\nabla(f \cdot s) = f \cdot F_\nabla(s)$. Hence show that F_∇ can be thought of as an element of $\Gamma(\Lambda^2 T^*M \otimes End(E))$.

(b) For any $X, Y \in \mathcal{T}^1(M)$, prove that

$$F_\nabla(X,Y)(s) := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,T]} s,$$

and hence prove that if $E = TM$ with the Levi-Civita connection, then F_∇ is simply the Riemann curvature endomorphism.

(c) Let $\{e_\alpha\}$ be a local frame for E, and let A_α^β be the local connection matrix of 1-forms. We define the local curvature matrix by $F_\nabla(e_\alpha) = \Omega_\alpha^{\beta \cdot}$ Prove *Cartan's second structural equation*

$$\Omega_\alpha^\beta = dA_\alpha^\beta - A_\alpha^\gamma \wedge A_\gamma^\beta.$$

(d) Let $\Gamma : [0,1] \times [0,1] \to M$ be a smooth map such that $\Gamma(s,0) = \Gamma(s,1) = p$ and $\Gamma(1,t) = p$. Let $P_{(s,t)} : E_p \to E_{\Gamma(s,t)}$ be the parallel translation along the curve $\Gamma_t(s) = \Gamma(s,t)$, and let $\hat{P}_{(s,t)} : E_p \to E_{\Gamma(s,t)}$ be the parallel translation first along $\gamma(t) := \Gamma(0,t)$ to $\gamma(t)$ and then along $\Gamma_t(s)$ to $\Gamma(s,t)$. If P_γ denotes the parallel translation along $\gamma(t)$, prove that

$$P_\gamma - id = \int\int_{[0,1]\times[0,1]} P_{(s,t)}^{-1} F_\nabla \hat{P}_{(s,t)}.$$

(e) Prove that ∇ is flat if and only if $\mathrm{Hol}_p^0(\nabla) = \{id\}$ for any $p \in M$.

4. Let $\Sigma = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2$ be the standard paraboloid with the induced metric. Prove that the Gauss curvature is given by $K(x,y,z) = (z+1)^{-2}$.

5. Let G be a Lie group with a bi-invariant metric g.

(a) Show that

$$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]],$$

whenever X, Y and Z are left-invariant vector fields.

(b) Show that all sectional curvatures are non-negative.

(c) If H is a Lie subgroup of G with the induced metric, show that H is totally geodesic.

(d) If H is connected, prove that it is flat in the induced metric if and only if it is Abelian.

6. Let (M,g) be a Riemannian manifold. We say that $(\tilde{M},\tilde{g})$ is (locally) *conformal* to (M,g) if there exists a (local) diffeomorphism (called a *conformal (local) diffeomorphism*) $\varphi : \tilde{M} \to M$ such that $\tilde{g} = e^u \varphi^* g$ for some $u \in C^\infty(M)$. We say that (M,g) is *locally conformally flat* if for every $p \in M$, there exists a neighbourhood U, and a conformal diffeomorphism of (U,g) onto an open set $V \subset \mathbb{R}^n$ with the Euclidean flat metric.

(a) Prove that if φ is a local conformal diffeomorphism from $(\tilde{M},\tilde{g})$ to (M,g), then the corresponding Weyl tensors satisfy $\varphi^* W = \tilde{W}$.

(b) If (M,g) is locally conformally flat, prove that $W \equiv 0$. **Note.** In a later assignment, we will show that the converse is also true. This is analogous to the characterisation of manifolds locally isometric to $\mathbb{R}^n$ by vanishing of the full curvature tensor.

7. Show that the catenoid $M \subset \mathbb{R}^3$ obtained by revolving $x = \cosh z$ around the z-axis is a minimal surface.

8. (a) For a Riemannian metric on a surface given locally by $g = h(dx^2 + dy^2)$, for some smooth strictly positive function, prove that the Gauss curvature has the formula

$$K = -\frac{1}{2h}\Big(\frac{\partial^2 \log h}{\partial x^2} + \frac{\partial^2 \log h}{\partial y^2}\Big).$$

(b) Use this to show that $\mathbb{H}_R^2$ has constant Gauss curvature $-1/R^2$.

(c) Let (M^n, g) be a Riemannian hypersurface in $\mathbb{R}^{n+1}$ with a unit normal vector field $\mathcal{V}$. We can think of $\mathcal{V}$ as a map $\mathcal{V} : M \to \mathbf{S}^n$, called the *Gauss map.* If $d\sigma$ is the standard round metric measure on $\mathbf{S}^n$, prove that

$$\mathcal{V}^* d\sigma = K dV_g,$$

where K is the Gaussian curvature of (M,g).