MA333: Assignment-5

(due 13 November, 2019)

October 31, 2019

Note. Please submit solutions to problems 2(b), 4, 5 and 6(c).

1. Suppose $\gamma : [a, b] \to M$ is a geodesic, and let $V \in T_{\gamma(a)}M$ and $W \in T_{\gamma(b)}M$. Prove that there exists a Jacobi field J along γ with $J(a) = V$ and $J(b) = W$ if and only if $\gamma(a)$ and $\gamma(b)$ are not conjugate along γ. Moreover, if such a Jacobi field exists, then it is unique.

2. (a) Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. Endow the product $M := M_1 \times M_2$ with the product metric $g := g_1 \oplus g_2$. Prove that

$$\mathrm{Rm}_g = \pi_1^*\mathrm{Rm}_{g_1} \oplus \pi_2^*\mathrm{Rm}_{g_2},$$

   where $\pi_i : M \to M_i$ are the projection maps.

   (b) Prove that the sectional curvatures of $S^2 \times S^2$ lie between $[0, 1]$.

3. Let (M, g) have non positive sectional sectional curvature. Let $J(t)$ be a Jacobi field along a unit speed geodesic. Prove that $f(t) = |J(t)|^2$ is convex as long as $J(t) \neq 0$. Hence prove that M cannot have any conjugate points.

4. Suppose (M^n, g) is a Riemannian manifold and $\mathcal{U}$ is a normal neighbourhood centred at $p \in M$.

   (a) Let $\gamma(t) = \exp_p(tv)$ with $|v| = 1$, and $J(t)$ be a Jacobi field along γ with $J(0) = 0$ and $\mathcal{D}_t J(0) = w$, where $|w| = 1$. Show that

$$|J(t)|^2 = t^2 - \frac{1}{3}Rm(v, w, w, v)t^4 + R(t),$$

   where $\lim_{t\to 0} t^{-4}R(t) = 0$.

   (b) Prove that with respect to the normal coordinates $\{x^i\}$ in $\mathcal{U}$, the following expansion holds

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{iklj}x^k x^l + O(|x|^3).$$

   Hint. Compute the first four derivatives of $|J(t)|^2$ in a different way, and use the first part.

   (c) Letting $S(p, r)$ and $B(p, r)$ denote the geodesic sphere and ball respectively of radius r, prove the asymptotic formulae as $r \to 0^+$

$$\frac{|B(p, r)|}{\omega_n r^n} = 1 - \frac{S(p)}{6(n + 2)}r^2 + O(r^3)$$

$$\frac{|S(p, r)|}{n\omega_n r^{n-1}} = 1 - \frac{S(p)}{6}r^2 + O(r^3),$$

   where $|S(p, r)|$ and $|B(p, r)|$ denote the volumes with respect to the induced metrics on $S(p, r)$ and $B(p, r)$ respectively, ω_n is the volume of the unit ball in $\mathbb{R}^n$ and S is the scalar curvature of (M, g).

(d) Now let $n = 2$ and denote by $A(p,r)$ and $C(p,r)$ the area and circumference respectively of geodesic r circles around p. Then prove that the Gauss curvature has the following geometric formulae

$$K(p) = 12 \lim_{r\to 0^+} \frac{\pi r^2 - A(p,r)}{\pi r^4} = 3 \lim_{r\to 0^+} \frac{2\pi r - C(p,r)}{\pi r^3}.$$

5. (Hessian of the distance function).

(a) Prove that if $\gamma : [0,b] \to M$ is a unit speed geodesic, and Γ is an admissible (possibly non-proper) one-parameter variation with $\partial_s \Gamma(0,\cdot) = W(\cdot)$ and $S := \partial_s \Gamma$, then

$$\frac{d}{ds^2}\Big|_{s=0} \mathcal{L}(\gamma_s) = \langle \mathcal{D}_s S(0,b), \dot\gamma(b)\rangle - \langle \mathcal{D}_s S(0,0), \dot\gamma(0)\rangle + \int_a^b \Big(|\mathcal{D}_t W|^2 - \langle \mathcal{D}_t W, \dot\gamma\rangle^2 - Rm(W,\dot\gamma,\dot\gamma,W)\Big).$$

Note. As mentioned in class, unlike the first variation formula for length, the above formula does not follow directly from the analogous formula for the energy. Also, unlike in the lectures, here we are not assuming that the variation is proper. You need to redo the complete calculation.

(b) Consider $f(x) := \frac{1}{2} d(p,x)^2$. For any $q \in M \setminus Cut(p)$ and $X_q \in T_q M$, let $\gamma_q : [0, f(q)] \to M$ be the unique unit speed minimal geodesic connecting p to q and J a Jacobi field along γ_q with $J(0) = 0$ and $J(f(q)) = X_q$ (This exists by Problem 1). Then prove that

$$\mathrm{Hess} f\ (X_q, X_q) = f(q)\langle J'(f(q)), X_q\rangle.$$

Note that $J'(t) := \nabla_{\dot\gamma(t)} J$. **Note.** There might be a negative sign in front of the RHS. If so, let me know.

Hint. Take a variation of γ induced by J, and apply the second variation formula above. Note that you will have to integrate by parts and worry about the boundary term at the right end-point.

(c) If (M,g) has non-positive sectional curvature, prove that the function $f(x)$ defined above is *convex* in the sense that for any geodesic γ, the function $f \circ \gamma$ is convex as a real valued function.

6. The *Chern-Gauss-Bonnet* theorem says that for any even dimensional closed manifold (M^n, g),

$$\chi(M) = \frac{1}{(2\pi)^m} \int_M \mathrm{Pf}(\Omega),$$

where $n = 2m$, $\chi(M)$ is the Euler characteristic, $\Omega_i{}^j := R_{kli}{}^j dx^k \wedge dx^l$ is the curvature form matrix, and

$$\mathrm{Pf}(\Omega) := \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} (-1)^{\mathrm{sgn}\sigma} \Omega_{\sigma(1)}^{\sigma(2)} \wedge \Omega_{\sigma(3)}^{\sigma(4)} \wedge \cdots \wedge \Omega_{\sigma(2m-1)}^{\sigma(2m)},$$

is the *Pfaffian* of the curvature.

(a) When $n = 2$, derive the *Gauss-Bonnet* theorem

$$\int_M K\, dA = 2\pi\chi(M),$$

where A is the area element induced by the metric g.

(b) When $n = 4$, prove that

$$\frac{1}{32\pi^2} \int_M \Big(|\mathrm{Rm}|^2 - 4|\mathrm{Rc}|^2 + S^2\Big)\, dV = \chi(M).$$

(c) Hence prove that if (M,g) is a four dimensional Einstein manifold, then $\chi(M) \geq 0$. Moreover, $\chi(M) = 0$ if and only if (M,g) is flat ie. locally isometric to $\mathbb{R}^4$.