MA333: Assignment-6

(due 28/11/2019)

Note. Submit solution to the entire problem. The aim of this problem is to prove a fundamental injectivity radius estimate and to relate volume ratios to injectivity radii under sectional curvature bounds. Let (M, g) be a complete Riemannian manifold.

1. In this first part, we wish to prove the following lemma:

   Lemma 1 (Klingenberg)**.** *Let $p \in M$ and $q \in Cut(p)$ such that $l := d(p,q) = d(p, Cut(p))$. If q is not conjugate to p along a minimizing geodesic from p to q, then there exists a unit speed smooth geodesic $\gamma : [0, 2l] \to M$ such that $\gamma(0) = \gamma(2l) = p$ and $\gamma(l) = q$.*

   Since $q \in Cut(p)$ but is not conjugate to p, there exist two distinct unit speed minimal geodesics $\gamma_1, \gamma_2 : [0, l] \to M$ with $\gamma_i(0) = p$ and $\gamma(l) = q$, where $l = d(p, q)$. It is enough to prove that $\gamma_1'(l) = -\gamma_2'(l)$ (This will automatically imply that $\gamma'(0) = -\gamma'(0)$ since by symmetry of the Cut locus and conjugate locus we also have $p \in Cut(q)$ and p os not conjugate to q). We proceed by contradiction. So suppose $\gamma_1'(l) \neq -\gamma_2'(l)$

   (a) Prove that there exists $w \in S_qM$ such that $\langle \gamma_i'(l), w \rangle < 0$ for $i = 1, 2$.

   (b) Let $\sigma(s) = \exp_q(sw)$. Prove that for $s << 1$, there exist one-parameter variations $\Gamma_i(s,t)$ such that for each $i = 1, 2$, $\gamma_{i,s} := \Gamma_i(s, \cdot)$ is a unit speed geodesic connecting p to $\sigma(s)$ and $\gamma_{i,0} = \gamma_i$.

   (c) Prove that for each $i = 1, 2$,
   $$\frac{d}{ds}\Big|_{s=0} \mathcal{L}(\gamma_{i,s}) < 0,$$
   and hence prove that for $s << 1$, $\sigma(s) \in Cut(p)$.

   (d) Prove that $d(p, \sigma(s)) < d(p, q)$ for $s << 1$, and hence conclude that there is a contradiction.

2. Hence prove that if $\sec_g \leq a^2$, then
   $$\mathrm{inj}_g(M) \geq \min\left(\frac{\pi}{a}, l\right),$$
   where l is the length of the shortest smooth closed geodesic contained in M.

3. Finally, we wish to prove the following fundamental result relating volume of small balls to injectivity radius lower bounds:

   Proposition 2. *There exists constants ρ and η depending only on the dimension n such that if (M^n, g) is a closed (ie. compact without boundary) Riemannian manifold with $|\sec_g| \leq a^2$, then there exists a point $p \in M$ with*
   $$\frac{|B(p,r)|}{r^n} \leq \frac{\eta}{r}\ \mathrm{inj}(M, g),$$
   for all $r \in (0, a\rho]$.

   (a) First, show that without loss of generality, one can assume $a = 1$, and that the theorem follows easily if $\mathrm{inj}_g(M) \geq \pi$. So for the rest of the proof we assume that $a = 1$ and $\mathrm{inj}_g(M) < \pi$. **Hint.** For the first part, rescale the metric. For the second part, use Bishop-Gromov.

(b) Let $\gamma : [0, l] \to M$ be a unit speed geodesic with $\gamma(0) = p$ and such that $l = \mathrm{inj}(M, g)$. Such a geodesic exists by Klingenberg lemma and our assumption that $\mathrm{inj}_g(M) < \pi$. Let $\{\dot\gamma(t), E_1(t), \cdots, E_{n-1}(t)\}$ be an orthonormal parallel frame of $T_{\gamma(t)}M$. For any small $r > 0$, consider the map $\Phi : [0, l] \times B_r(0) \to M$ defined by

$$\Phi(t, V^1, \cdots, V^{n-1}) = \exp_{\gamma(t)} \Big(\sum_{i=1}^{n-1} V^i E_i(t) \Big).$$

Here $B_r(0)$ is the Euclidean ball of radius r in $\mathbb{R}^{n-1}$, and we pick ε small enough so that the above map is a diffeomorphism. Prove that if $\sec_g \geq -1$, then there exist constants $\rho = \rho(n)$ such that if $r < \rho$, then Φ is an immersion and $|\det D\Phi| \leq 2^n$ on $[0, l] \times B_\varepsilon(0)$. **Hint.** Calculate the determinant in terms of Jacobi fields and use Rauch's theorem.

(c) Prove that

$$|B(p, r)| \leq |\Phi([0, l] \times B_r(0))| \leq 2^n l \omega_{n-1} r^{n-1},$$

where ω_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$, and hence complete the proof of the Proposition. **Hint.** You can use use change of variables even though Φ might not be a diffeomorphism. The cost is that one gets an inequality instead of an equality when changing variables, but this inequality is in the right direction.

Historical note. One of Perelman's first major breakthrough on Ricci flows was to prove that for a finite time Ricci flows with bounded scalar curvature, the volume ratio of small balls has a uniform lower bound. Together with the above bound, one could then apply Hamilton's previous work (which relied on injectivity bounds) to study blow-up limits (Note that to study blow-up limits, the idea is to rescale the flow so that the metrics have bounded curvature, and since the volume ratio is invariant under rescaling, the above estimate gives a lower bound on the injectivity radius). This was a well known roadblock in proving Poincare conjecture via Ricci flow, and was done away with by Perelman in the first few pages of his first paper (of three) on solving the uniformization conjecture.