Kähler-Einstein metrics on Fano manifolds

Ved Datar

UC Berkeley

Jan 02, 2018
Outline

1. Introduction: Some history and the main theorem
2. $K$-stability
3. Outline of the proof
4. What next?
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2. $K$-stability

3. Outline of the proof

4. What next?
**Uniformization Theorem.**

*Theorem (Uniformization theorem)*

- Any compact Riemann surface admits a metric of constant Gauss curvature.
- Given any oriented compact 2-d Riemannian manifold \((M, g_0)\), there exists a metric \(g = e^{2\varphi}g_0\) with constant Gauss curvature.
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- Given any oriented compact 2-d Riemannian manifold \((M, g_0)\), there exists a metric \(g = e^{2\varphi} g_0\) with constant Gauss curvature.
For a surface with metric (in isothermal coordinates)

\[ ds^2 = h(dx^2 + dy^2), \]

the Gauss curvature form is given by

\[ KdA = -\sqrt{-1}\frac{\partial^2 \log h}{\partial z \partial \bar{z}}dz \wedge d\bar{z}, \]

where \( z = x + \sqrt{-1}y \). By Gauss-Bonnet,

\[ \int_M K dA = 2\pi \chi(M). \]

So depending on the sign of \( \chi(M) \), a compact Riemann surface admits a metric of constant Gauss curvature \( K = \pm 1 \) or \( K = 0 \).

Remark

A purely PDE proof of the case \( K = 1 \) (ie. \( M = S^2 \)) is the hardest. This is a harbinger of things to come!
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Let $M^n$ be a compact complex manifold with $\dim_{\mathbb{C}} M = n$.

A Riemannian metric $g$ is called Kähler if there are local coordinates $(x^1, \cdots, x^{2n})$ in which

$$g_{jk} = \delta_{jk} + O(|x|^2),$$

and such that for $j = 1, \cdots, n$,

$$z^j = x^j + \sqrt{-1}x^{n+j}$$

are local holomorphic coordinates.

One can associate a $(1, 1)$ form in the following way - If $J$ denotes the canonical complex structure

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^{n+j}}, \quad J^2 = -\text{id},$$

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$$\omega(\cdot, \cdot) = g(J\cdot, \cdot).$$

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Properties of the Kähler form

- $\omega$ is a closed, real form (i.e. $\bar{\omega} = \omega$), and so represents a cohomology class in $H^2(M, \mathbb{R})$.

- Locally we can write
  
  $$\omega = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta,$$

  where $g_{\alpha\bar{\beta}} = g \left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right)$. Then the matrix $\{g_{\alpha\bar{\beta}}\}$ is a positive definite Hermitian matrix.

- Conversely, given such a form $\omega$, $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ defines a Riemannian metric.

- Any class in $H^2(M, \mathbb{R})$ which contains a Kähler metric, is called a Kähler class. The set of Kähler classes $\mathcal{K} \subset H^2(M, \mathbb{R})$ is an open convex cone.

- ((\partial \bar{\partial}) Lemma) If $[\omega_1] = [\omega_2]$, then there exists a $\varphi \in C^\infty(M, \mathbb{R})$ such that
  
  $$\omega_2 = \omega_1 + \sqrt{-1} \partial \bar{\partial} \varphi.$$

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Examples

- Any compact Riemann surface with $\omega$ being a volume form. Then since $n = 1$, $d\omega = 0$ is trivially true.
- Complex projective space $\mathbb{P}^N$ with the Fubini study metric given in homogenous coordinates $[\xi_0, \cdots, \xi_N]$ by

$$\omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log \left(|\xi_0|^2 + \cdots |\xi_N|^2\right).$$

When $n = 1$, $\mathbb{P}^1 = S^2$, and the Fubini-Study metric is the usual round metric.
- Any non-singular sub-variety $X \subset \mathbb{P}^N$, with $\omega$ given by restricting $\omega_{FS}$ to $X$. 
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Curvature

- The **Ricci form** of the Kähler metric $\omega$ is defined by

$$\rho_\omega = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{\alpha \bar{\beta}}).$$

Note that $d \rho_\omega = 0$.

- The Riemannian Ricci curvature is then given by

$$\text{Ric}(\cdot, \cdot) = \rho_\omega (\cdot, J\cdot).$$

**Question (Calabi, 1950s)**

*Given a Kähler manifold $M$, when does it admit a metric of constant Ricci curvature. More precisely, when is there a Kähler metric $\omega$ such that

$$\rho_\omega = \lambda \omega,$$

for some $\lambda \in \mathbb{R}$. Such an $\omega$ is called a **Kähler-Einstein (KE)** metric.*
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\frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}.
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If $h$ is a hermitian metric then the curvature $F_h = -\partial \bar{\partial} \log h$ is a global closed $(1, 1)$ form, and the first Chern class $c_1(M) \in H^2(M, \mathbb{R})$ is defined by
\[
c_1(M) := c_1(K_M^{-1}) := \frac{\sqrt{-1}}{2\pi} [F_h].
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So if $\omega$ is KE, then
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(Yau [8], '78) If $c_1(M) = 0$, then there exists a unique Ricci flat metric $\omega$ in every Kähler class.

(Yau, Aubin [1], '78) If $c_1(M) < 0$, then there exists a unique KE metric in $-2\pi c_1(M)$.

(Chen-Donaldson-Sun [2], 2012) If $c_1(M) > 0$ (ie. $M$ is Fano), there exists KE in $2\pi c_1(M)$ if $M$ is $K$-stable. Converse due to Tian, Berman etc.
Existence results

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Introduction: Some history and the main theorem

Some remarks

- Fano case an instance of **Kobayashi-Hitchin correspondence**

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\{ \text{Existence of Canonical metrics} \} & \iff \{ \text{Algebro-geometric Stability} \} \\
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- Examples - Narasimhan-Seshadri, Donaldson-Yau-Uhlenbeck, Kähler-Einstein metrics on $S^2$ with cone angles $< 2\pi, \cdots$.
- Unfortunately $K$-stability is notoriously difficult to check. eg. Even in manifolds with large symmetry groups (eg. Toric)
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Main theorem

Main Theorem (D.-Szekelyhidi [4], 2015)

Let $G \subset \text{Aut}(M)$ be a compact group. Then $M$ admits a KE if it is $G$-equivariantly $K$-stable with respect to special degenerations.

Remarks

- We actually prove a much more general theorem on existence of Kähler-Ricci solitons.
- The theorem can recover some older results (e.g., KE metrics on toric manifolds), and has also led to the discovery of new KE manifolds.
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2 $K$-stability

3 Outline of the proof

4 What next?
An analytic obstruction: Futaki invariant

- Let $M$ be a Fano manifold, that is $c_1(M) > 0$, and let $\omega \in 2\pi c_1(M)$ be a Kähler metric.
- Since $[\rho_\omega] = [\omega]$, there exists an $h \in C^\infty(M, \mathbb{R})$, called the Ricci potential, such that
  $$\rho_\omega = \omega + \sqrt{-1}\partial\bar{\partial}h.$$ 
- Let $\eta(M) = \{\text{holomorphic vector fields on } M\}$, and define $\text{Fut} : \eta(M) \to \mathbb{C}$ by
  $$\text{Fut}(M, \xi) = -\frac{1}{V} \int_M \xi(h) \frac{\omega^n}{n!},$$
  where $V = \int_M \frac{\omega^n}{n!}$ is the volume of the manifold.
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Example

\( Bl_p \mathbb{P}^2 \) does not admit a KE. The Futaki does not vanish!

Question

Is this the only obstruction?

Answer

(Tian '97) NO! Examples of Kähler three-folds (so called Mukai-Umemera manifolds) with \( \eta(M) = \{0\} \) and yet not admitting any KE.

In general one needs to allow the manifold to degenerate to a possibly singular variety. Key point.

- (Ding-Tian) The Futaki invariant can be defined on “sufficiently nice” singular varieties.
- (Donaldson) Algebro-geometric definition using Riemann-Roch.
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Futaki invariant of a degeneration

- Since \( c_1(M) > 0 \), \( K_M^{-1} \) is ample, so there is a Kodaira embedding \( M \hookrightarrow \mathbb{P}^N \) for some large \( N \).

- A special degeneration is a one-parameter subgroup \( \lambda : \mathbb{C}^* \to GL(N, \mathbb{C}) \) generated by the holomorphic vector field \( \xi \in \mathfrak{gl}(N, \mathbb{C}) \), such that the flat limit
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  W = \lim_{t \to 0} \lambda(t) \cdot M
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  is a normal \( \mathbb{Q} \)-Fano variety. Note that \( \xi \) is tangential to \( W \).

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Definition

$M$ is called $K$-semistable if for any Kodaira embedding, and any special degeneration $\lambda$, $\text{Fut}(M, \lambda) \geq 0$. It is called $K$-stable if in addition, $\text{Fut}(M, \lambda) = 0$ if and only if there exists a $A \in \text{GL}(N, \mathbb{C})$ such that $W = A \cdot M$.

If $G \subset \text{Aut}(M)$, then $G$ acts naturally on $K_M^{-1}$, and hence can be identified as a subgroup $G \subset \text{GL}(N, \mathbb{C})$ for any Kodaira embedding $M \hookrightarrow \mathbb{P}^N$.

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We say $\lambda(t)$ is a $G$-equivariant special degeneration, if $\lambda(t) : \mathbb{C}^* \rightarrow \text{GL}(N, \mathbb{C})^G$, and we then analogously define $G$-equivariant $K$-(semi)stability with $A \in \text{GL}(N, \mathbb{C})^G$.

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**Figure:** *K*-semistable.  
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- Consider the embedding

\[ \mathbb{P}^1 \hookrightarrow \mathbb{P}^2 \]

\[ [\xi_0, \xi_1] \mapsto [\xi_0^2, \xi_0 \xi_1, \xi_1^2]. \]

The image \( M \) is the conic \( y^2 - xz = 0 \) in \( \mathbb{P}^2 \).

- If we let

\[ \lambda(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

then \( M_t = \lambda(t) \cdot M \), is given by the conic \( t^2 y^2 - t^{-1} xz = 0 \), and so the flat limit \( W = (xz = 0) \).

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Main Theorem (D.-Szekelyhidi [4], 2015)

Let \( G \subseteq \text{Aut}(M) \) be a compact group. Then \( M \) admits a KE if it is \( G \)-equivariantly \( K \)-stable with respect to special degenerations.

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Toric manifolds. Recall that \( M \) is toric if there is an effective holomorphic action of \((\mathbb{C}^*)^n\) with a free, open, dense orbit.

Theorem (Wang-Zhu, 2003)

\( M \) admits KE \iff the classical Futaki invariant vanishes.

Proof.

\( \Rightarrow \) is true in general. For converse, let \( G = (S^1)^n \). One can show that all equivariant degenerations are trivial, and since the classical Futaki is zero by hypothesis, \( M \) is equivariantly \( K \)-stable. By main theorem, it admits KE.
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Concretely, \( Q \subset \mathbb{P}^4 \) be a quadric, and let \( M \) be the blow-up along a conic. This admits a KE, and the only known proof is via the main theorem.
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Let $\alpha \in 2\pi c_1(M)$ be a Kähler metric invariant under $G$, and consider the equation

$$Ric(\omega_t) = t\omega_t + (1 - t)\alpha. \quad (3.1)$$

Here, and henceforth, we will be using the notation $Ric$ to also denote the Ricci form. Let

$$I = \{ t \in [0, 1] \mid (3.1) \text{ has a solution} \}.$$  

To show $1 \in I$, it is enough to show

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Let $\alpha \in 2\pi c_1(M)$ be a Kähler metric invariant under $G$, and consider the equation

$$Ric(\omega_t) = t\omega_t + (1 - t)\alpha. \quad (3.1)$$

Here, and henceforth, we will be using the notation $Ric$ to also denote the Ricci form. Let

$$I = \{ t \in [0, 1] \mid (3.1) \text{ has a solution} \}.$$ 

To show $1 \in I$, it is enough to show

- $0 \in I$,
- $I$ is open in $[0, 1]$,
- $I$ is closed.
An interjection: The Donaldson continuity method

- Chen-Donaldson-Sun consider the following continuity method instead

\[ \text{Ric}(\omega_\beta) = \beta \omega_\beta + \frac{1 - \beta}{m} [D]. \]

Here \( D \) is some smooth co-dimension 1 sub-variety (smooth divisor) in \( |-mK_M| \).

- The metrics \( \omega_\beta \) have conical singularities along the divisor \( D \).
- Disadvantages -
  - (Song-Wang) NO smooth \( G \) invariant divisors unless \( G \) is finite.
  - (Song-Wang, D.-Guo-Song-Wang) If we relax \( D \) to be only simple-normal-crossing, then the above equation in most cases does not have a solution.

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PDE aspect

We can recast the equation (3.1) into the following complex Monge-Ampere equation:

\[
\begin{cases}
(\alpha + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{-t\varphi_t + h}\alpha^n \\
\omega_t := \alpha + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0,
\end{cases}
\]

(3.2)

where \( h \) is the Ricci potential of the form \( \alpha \).

- At \( t = 0 \), we need to solve \( \text{Ric}(\omega_0) = \alpha \), where \( \alpha \) is a given positive form. This is precisely the Calabi conjecture solved by Yau [8].
- (Openness) The linearization of the equation is given by

\[
\Delta\omega_t + t.
\]

If \( t < 1 \), then \( \text{Ric}(\omega_t) > t\omega_t \), and a Bochner-type argument shows that \( \lambda_1(\Delta\omega_t) > t \). Hence the operator above is invertible, and implicit function theorem \( \implies \) openness.
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(Closedness) From a priori estimates and Arzela-Ascoli.

Proposition (Aubin and Yau, 1970s)

For every \( A \) and \( k \in \mathbb{N} \), there exists constants \( C_k \) such that

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\| \varphi_t \|_{C^0} \leq A \implies \| \varphi_t \|_{C^k} \leq C_k.
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So to complete the proof we “only” need uniform \( C^0 \) estimates. This is where K-stability comes in.

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Gromov-Hausdorff convergence

- **The metrics satisfy**
  1. $\text{Vol}(M, g_k) = V := \frac{(2\pi)^n}{n!} c_1(M)^n$.
  2. $\text{Ric}(g_k) \geq t_0 g_k$.
  3. (Meyers') $\text{diam}(M, g_k) \leq \pi \sqrt{\frac{2n-1}{t_0}}$.
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     $$|B_r| \geq \kappa r^{2n}.$$ 

- (Gromov) The sequence of Riemannian manifolds $(M, g_k) \xrightarrow{G-H} (Z, d)$, where $(Z, d)$ is a compact metric length space.

- $Z$ is a candidate for the central fiber of a destabilizing special degeneration.

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The following theorem was conjectured by Yau, and made precise, and proved by Tian when $n = 2$.

Theorem (Donaldson-Sun (for KE) [5], Szekelyhidi (in the present context) [7])

There exists a uniform $m$, and embeddings $F_k : M \hookrightarrow \mathbb{P}^N$ by sections of $H^0(M, K_M^{-m})$ with the following properties

1. $F_k$ are uniformly Lipschitz.
2. $F_k(M)$ converge to a $\mathbb{Q}$-Fano normal flat limit $W$, and the maps $F_k$ converge to a homeomorphism $F : Z \to W$.
3. (partial $C^0$ estimate) There exists a uniform constant $C$ such that

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Metric geometry to algebraic geometry: Partial $C^0$-estimate

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2. $F_k(M)$ converge to a $\mathbb{Q}$-Fano normal flat limit $W$, and the maps $F_k$ converge to a **homeomorphism** $F : Z \rightarrow W$.
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The following theorem was conjectured by Yau, and made precise, and proved by Tian when \( n = 2 \).

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After passing to a subsequence, $\rho_k \to g \in GL^G$.

**Proof of main theorem.**

Proposition $\implies$

$$\frac{1}{m} \rho_k^* \omega_{FS} - \frac{1}{m} \omega_{FS} = \sqrt{-1} \partial \overline{\partial} \nu_k$$

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Outline of the proof

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- The currents $\rho_k(\omega_k)$ converge weakly to a weak current $\omega_T$ supported on $W$ solving the twisted KE equation on $W$ in the weak sense -

$$\text{Ric}(\omega_T) = T\omega_T + (1 - T)\beta.$$  

Consequences -

1. “Aut($W, \beta$)” is reductive.
2. The “twisted Futaki invariant”

$$\text{Fut}_{(1-T)\beta}(W, w) = 0$$

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- $W$ is in the $GL^G$ orbit closure but might not be accessible by $\mathbb{C}^*$.
- If one could embed all such pairs $(W, \beta)$ into a large finite dimensional projective, then since the stabilizer is reductive, one could use Luna slice theorem.
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Main argument

1. Choose generic hyperplanes \( \{V_i\}_{i=1}^d \) such that
   - Passing to a subsequence \( \rho_k(V_i) \) converges to hyperplane \( H_i \) for each \( i \).
   - \( \beta \approx \frac{1}{d} \sum_{i=1}^d [W \cap H_i] \).
   - \( \text{aut}(W, \beta) = \text{aut}(W, W \cap H_1, \ldots, W \cap H_d) \).
2. Then \( \text{aut}(W, W \cap H_1, \ldots, W \cap H_d) \) is reductive and
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3. Luna slicing \( \implies \exists \lambda(t) : \mathbb{C}^* \to GL^G \) generated by a vector field \( w \), and a fixed \( g \in GL^G \) such that
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Then $\text{aut}(W, W \cap H_1, \cdots, W \cap H_d)$ is reductive and

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If $\theta_w$ is the Hamiltonian of $w$, then

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$K$-stability $\implies$ it should be equality, and hence $w = 0$.

Since $w = 0$, the degeneration is trivial, and so

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$$\text{Fut}_{(1-T)\beta}(W, w) = \text{Fut}(W, w) - (1 - T) \frac{n}{V} \int_W \theta_w(\beta - \omega_{FS}) \omega_{FS}^{n-1}.$$ 

Using the fact that $\text{Fut}_{(1-T)\beta}(W, w) = 0$, a calculation shows that

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$K$-stability $\implies$ it should be equality, and hence $w = 0$.

Since $w = 0$, the degeneration is trivial, and so

$$(W, W \cap H_1, \ldots, W \cap H_d) = g \cdot (M, M \cap V_1, \ldots, M \cap V_d).$$

Recall that $H_i = \lim_{k \to \infty} \rho_k(V_i)$.

Since $V_i$ are generic, a simple argument now shows that $\rho_k \to g$. \hfill \blacksquare
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1 Introduction: Some history and the main theorem

2 $K$-stability

3 Outline of the proof

4 What next?
Extremal metrics

- Let $L \to M$ be an ample line bundle.
- (Calabi, 1980s) The critical points of the functional
  \[ Ca(\omega) = \int_M \|Rm(\omega)\|^2, \]
  as $\omega$ varies over Kähler metrics in the fixed co-homology class $2\pi c_1(L)$ are called extremal metrics.
- The Euler-Lagrange equation says that
  \[ \bar{\partial} \nabla^{1,0} s_\omega = 0, \]
  where $s_\omega$ is the scalar curvature.
- In particular constant scalar curvature Kähler metrics (cscK), and hence Kähler-Einstein metrics, are automatically extremal.
- It is expected that existence is again related to certain stability, called relative $K$-stability (or some refinement).
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Perturbation problems

Suppose $M$ admits an extremal metric $\omega$. If $p \in M$, and $\pi : Bl_p(M) \to M$ is the blow-up with exceptional divisor $E$, then it is known that $L_\varepsilon = \pi^*\omega - \varepsilon^2[E]$ is Kähler for $\varepsilon << 1$.

**Question**

If $(Bl_p(M), L_\varepsilon)$ is relatively $K$-stable, does it admit an extremal metric?

- (Szekelyhidi) If $n > 2$, the answer is affirmative for cscK metrics.
- When $n = 2$, I have recently made some progress [3], but an optimal result is still missing.
- The problem for extremal metrics is completely open in all dimensions.
- The key difficulty is in relating relative $K$-stability of blow-ups $Bl_p(M)$ to the relative GIT stability of the point $p$. 
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Ved Datar (UC Berkeley)
What next?

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Convergence of cscK manifolds and partial $C^0$ estimate

- Sequences of cscK manifolds might collapse.

**Question**

Is there a uniform partial $C^0$ estimate along a sequence of non-collapsed, cscK metrics on Kähler manifolds with uniform bounds on the total volume, and Calabi energy?

- For $n = 2$, there is an optimal convergence result due to Anderson and Tian-Viaclovsky.
- For $n > 2$, the convergence result assumes $L^{n/2}$ bound on $||Rm||$, which is not useful, since $Ca(\omega)$ involves an $L^2$ bound.
- What if there is collapsing? Is there a Cheeger-Tian type $\varepsilon$-regularity result for $n = 2$ (small $||Rm||_{L^2} \implies$ control on $||Rm||_{L^\infty}$)?
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Thank You for your attention!
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