Kähler-Einstein metrics on Fano manifolds

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Outline

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1. Introduction: Some history and the main theorem
2. K-stability
3. Outline of the proof
4. What next?

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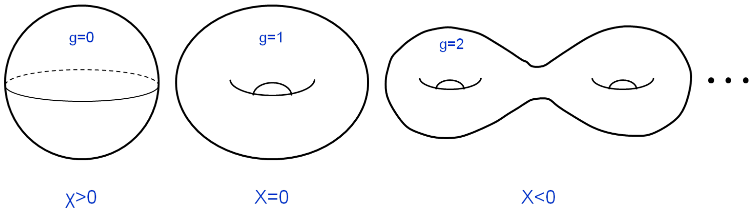
Uniformization Theorem.



Theorem (Uniformization theorem)

- *Any compact Riemann surface admits a metric of constant Gauss curvature.*
- *Given any oriented compact 2-d Riemannian manifold (M, g_0), there exists a metric $g = e^{2\varphi} g_0$ with constant Gauss curvature.*

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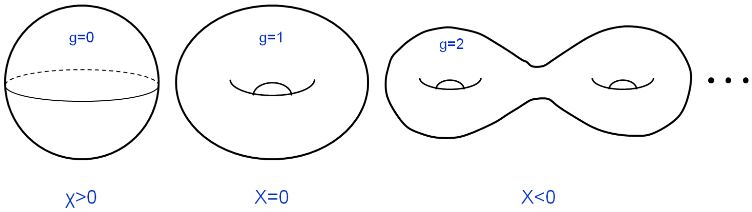

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- *Given any oriented compact 2-d Riemannian manifold* (M, g_0)*, there exists a metric* $g = e^{2\varphi} g_0$ *with constant Gauss curvature.*

For a surface with metric (in isothermal coordinates)

$$ds^2 = h(dx^2 + dy^2),$$

the Gauss curvature form is given by

$$KdA = -\sqrt{-1}\frac{\partial^2 \log h}{\partial z \partial \bar{z}} dz \wedge d\bar{z},$$

where $z = x + \sqrt{-1}y$. By Gauss-Bonnet,

$$\int_M K\, dA = 2\pi\chi(M).$$

So depending on the sign of $\chi(M)$, a compact Riemann surface admits a metric of constant Gauss curvature $K = \pm 1$ or $K = 0$.

Remark

A purely PDE proof of the case $K = 1$ (ie. $M = S^2$) is the hardest. This is a harbinger of things to come!

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Kähler manifolds

- Let M^n be a compact complex manifold with $\dim_{\mathbb{C}} M = n$.
- A Riemannian metric g is called Kähler if there are local coordinates $(x^1, \cdots, x^{2n})$ in which
$$g_{jk} = \delta_{jk} + O(|x|^2),$$
and such that for $j = 1, \cdots, n$,
$$z^j = x^j + \sqrt{-1}x^{n+j}$$
are local holomorphic coordinates.
- One can associate a $(1,1)$ form in the following way - If J denotes the canonical complex structure
$$J\Big(\frac{\partial}{\partial x^j}\Big) = \frac{\partial}{\partial x^{n+j}},\ J^2 = -\mathrm{id},$$
then we define the Kähler form ω by
$$\omega(\cdot,\cdot) = g(J\cdot,\cdot).$$
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Properties of the Kähler form

- ω is a closed, real form (ie. $\bar{\omega} = \omega$), and so represents a cohomology class in $H^2(M, \mathbb{R})$.
- Locally we can write
$$\omega = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta},$$
where $g_{\alpha\bar{\beta}} = g\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \bar{z}^{\beta}}\right)$. Then the matrix $\{g_{\alpha\bar{\beta}}\}$ is a positive definite Hermitian matrix.
- Conversely, given such a form ω, $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ defines a Riemannian metric.
- Any class in $H^2(M, \mathbb{R})$ which contains a Kähler metric, is called a Kähler class. The set of Kähler classes $\mathcal{K} \subset H^2(M, \mathbb{R})$ is an open convex cone.
- ($\partial\bar{\partial}$ Lemma) If $[\omega_1] = [\omega_2]$, then there exists a $\varphi \in C^{\infty}(M, \mathbb{R})$ such that
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Examples

- Any compact Riemann surface with ω being a volume form. Then since $n = 1$, $d\omega = 0$ is trivially true.
- Complex projective space $\mathbb{P}^N$ with the Fubini study metric given in homogenous coordinates $[\xi_0, \cdots \xi_N]$ by

$$\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log\left(|\xi_0|^2 + \cdots |\xi_N|^2\right).$$

  When $n = 1$, $\mathbb{P}^1 = S^2$, and the Fubini-Study metric is the usual round metric.
- Any non-singular sub-variety $X \subset \mathbb{P}^N$, with ω given by restricting ω_{FS} to X.

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Curvature

- The Ricci form of the Kähler metric ω is defined by

$$\rho_\omega = -\sqrt{-1}\partial\overline{\partial}\log\det(g_{\alpha\bar{\beta}}).$$

Note that $d\rho_\omega = 0$.

- The Riemannian Ricci curvature is then given by

$$\mathrm{Ric}(\cdot,\cdot) = \rho_\omega(\cdot, J\cdot).$$

Question (Calabi, 1950s)

Given a Kähler manifold M, when does it admit a metric of constant Ricci curvature. More precisely, when is there a Kähler metric ω such that

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for some $\lambda \in \mathbb{R}$. Such an ω is called a Kähler-Einstein (KE) metric.

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Topological obstruction

- Recall that the anti-canonical bundle K_M^{-1} is the line bundle locally generated by

$$\frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}.$$

- If h is a hermitian metric then the curvature $F_h = -\partial\bar{\partial} \log h$ is a global closed $(1,1)$ form, and the first Chern class $c_1(M) \in H^2(M, \mathbb{R})$ is defined by

$$c_1(M) := c_1(K_M^{-1}) := \frac{\sqrt{-1}}{2\pi}[F_h].$$

- Given Kähler metric ω, a natural hermitian metric on K_M^{-1} given by

$$\left|\left|\frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}\right|\right|^2 = \det g_{\alpha\bar{\beta}},$$

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Existence results

- (Yau [8], '78) If $c_1(M) = 0$, then there exists a unique Ricci flat metric ω in every Kähler class.
- (Yau, Aubin [1], '78) If $c_1(M) < 0$, then there exists a unique KE metric in $-2\pi c_1(M)$.
- (Chen-Donaldson-Sun [2], 2012) If $c_1(M) > 0$ (ie. M is Fano), there exists KE in $2\pi c_1(M)$ if M is K-stable. Converse due to Tian, Berman etc.

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Some remarks

- Fano case an instance of Kobayashi-Hitchen correspondence

$$\left\{\begin{matrix}\text{Existence of}\\ \text{Canonical metrics}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\text{Algebro-geometric}\\ \text{Stability}\end{matrix}\right\}$$

- Examples - Narasimhan-Seshadri, Donaldson-Yau-Uhlenbeck, Kähler-Einstein metrics on S^2 with cone angles $< 2\pi, \cdots$.
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Main theorem

Main Theorem (D.-Szekelyhidi [4], 2015)

Let $G \subset Aut(M)$ be a compact group. Then M admits a KE if it is G-equivariantly K-stable with respect to special degenerations.

Remarks

- *We actually prove a much more general theorem on existence of Kähler-Ricci solitons.*
- *The theorem can recover some older results (eg. KE metrics on toric manifolds), and has also led to the discovery of new KE manifolds.*
- *We need to use a completely different proof from that of CDS.*

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An analytic obstruction : Futaki invariant

- Let M be a Fano manifold, that is $c_1(M) > 0$, and let $\omega \in 2\pi c_1(M)$ be a Kähler metric.
- Since $[\rho_\omega] = [\omega]$, there exists an $h \in C^\infty(M, \mathbb{R})$, called the Ricci potential, such that

$$\rho_\omega = \omega + \sqrt{-1}\partial\bar{\partial}h.$$

- Let $\eta(M) = \{\text{hol. vector fields on } M\}$, and define $\mathrm{Fut} : \eta(M) \to \mathbb{C}$ by

$$\mathrm{Fut}(M, \xi) = -\frac{1}{V}\int_M \xi(h)\frac{\omega^n}{n!},$$

  where $V = \int_M \frac{\omega^n}{n!}$ is the volume of the manifold.
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Example

$Bl_p\mathbb{P}^2$ does not admit a KE. The Futaki does not vanish!

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Is this the only obstruction?

Answer

(Tian '97) NO! Examples of Kähler three-folds (so called Mukai-Umemera manifolds) with $\eta(M) = \{0\}$ and yet not admitting any KE.

In general one needs to allow the manifold to degenerate to a possibly singular variety.
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- (Ding-Tian) The Futaki invariant can be defined on "sufficiently nice" singular varieties.
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Futaki invariant of a degeneration

- Since $c_1(M) > 0$, K_M^{-1} is ample, so there is a Kodaira embedding $M \hookrightarrow \mathbb{P}^N$ for some large N.

- A special degeneration is a one-parameter subgroup $\lambda : \mathbb{C}^* \to GL(N, \mathbb{C})$ generated by the holomorphic vector field $\xi \in \mathfrak{gl}(N, \mathbb{C})$, such that the flat limit

$$W = \lim_{t \to 0} \lambda(t) \cdot M$$

is a normal $\mathbb{Q}$-Fano variety. Note that ξ is tangential to W.

- We define the Futaki invariant for this degeneration by

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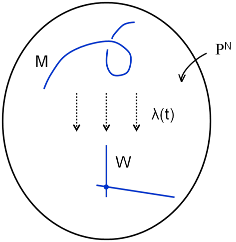

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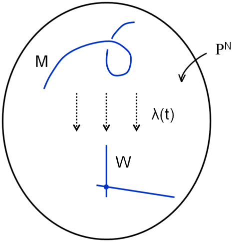

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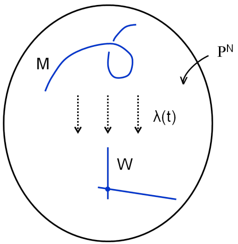

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Definition

M is called K-semistable if for any Kodaira embedding, and any special degeneration λ, $\mathrm{Fut}(M,\lambda) \geq 0$. *It is called K-stable if in addition,* $\mathrm{Fut}(M,\lambda) = 0$ *if and only if there exists a* $A \in GL(N,\mathbb{C})$ *such that* $W = A \cdot M$.

If $G \subset Aut(M)$, then G acts naturally on K_M^{-1}, and hence can be identified as a subgroup $G \subset GL(N,\mathbb{C})$ for any Kodaira embedding $M \hookrightarrow \mathbb{P}^N$.

Definition

We say $\lambda(t)$ *is a G-equivariant special degeneration, if* $\lambda(t) : \mathbb{C}^* \to GL(N,\mathbb{C})^G$, *and we then analogously define G-equivariant K-(semi)stability with* $A \in GL(N,\mathbb{C})^G$.

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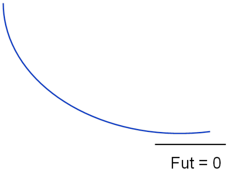

Figure: K-semistable.

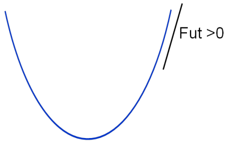

Figure: K-stable.

Example

- Consider the embedding
$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$$
$$[\xi_0, \xi_1] \mapsto [\xi_0^2, \xi_0\xi_1, \xi_1^2].$$
The image M is the conic $y^2 - xz = 0$ in $\mathbb{P}^2$.
- If we let
$$\lambda(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
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Applications of the main theorem

Main Theorem (D.-Szekelyhidi [4], 2015)

Let $G \subset Aut(M)$ be a compact group. Then M admits a KE if it is G-equivariantly K-stable with respect to special degenerations.

Example

Toric manifolds. *Recall that M is toric if there is an effective holomorphic action of $(\mathbb{C}^*)^n$ with a free, open, dense orbit.*

Theorem (Wang-Zhu, 2003)

M admits KE $\iff$ the classical Futaki invariant vanishes.

Proof.

$\implies$ is true in general. For converse, let $G = (S^1)^n$. One can show that all equivariant degenerations are trivial, and since the classical Futaki is zero by hypothesis, M is equivariantly K-stable. By main theorem, it admits KE. □

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Continuity method

Let $\alpha \in 2\pi c_1(M)$ be a Kähler metric invariant under G, and consider the equation

$$Ric(\omega_t) = t\omega_t + (1-t)\alpha. \tag{3.1}$$

Here, and henceforth, we will be using the notation Ric to also denote the Ricci form. Let

$$I = \{t \in [0,1] \mid (3.1) \text{ has a solution}\}.$$

To show $1 \in I$, it is enough to show

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$$Ric(\omega_\beta) = \beta\omega_\beta + \frac{1-\beta}{m}[D].$$

Here D is some smooth co-dimension 1 sub-variety (smooth divisor) in $|-mK_M|$

- The metrics ω_β have conical singularities along the divisor D.
- Disadvantages -
  - (Song-Wang) NO smooth G invariant divisors unless G is finite.
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- Chen-Donaldson-Sun consider the following continuity method instead

$$Ric(\omega_\beta) = \beta\omega_\beta + \frac{1-\beta}{m}[D].$$

  Here D is some smooth co-dimension 1 sub-variety (smooth divisor) in $|-mK_M|$
- The metrics ω_β have conical singularities along the divisor D.
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  - (Song-Wang) NO smooth G invariant divisors unless G is finite.
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PDE aspect

We can recast the equation (3.1) into the following complex Monge-Ampere equation :

$$\begin{cases} (\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = e^{-t\varphi_t + h}\alpha^n \\ \omega_t := \alpha + \sqrt{-1}\partial\overline{\partial}\varphi_t > 0, \end{cases} \tag{3.2}$$

where h is the Ricci potential of the form α.

- At $t = 0$, we need to solve $Ric(\omega_0) = \alpha$, where α is a given positive form. This is precisely the Calabi conjecture solved by Yau [8].
- (Openness) The linearization of the equation is given by

$$\Delta_{\omega_t} + t.$$

If $t < 1$, then $Ric(\omega_t) > t\omega_t$, and a Bochner-type argument shows that $\lambda_1(\Delta_{\omega_t}) > t$. Hence the operator above is invertible, and implicit function theorem $\implies$ openness.

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Proposition (Aubin and Yau, 1970s)

For every A and $k \in \mathbb{N}$, there exists constants C_k such that

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From now on, let $t_k \to T$ where $t_k \in I$. We need to show that $T \in I$. For ease of notation, let $\omega_k = \omega_{t_k} = \alpha + \sqrt{-1}\partial\bar{\partial}\varphi_k$, and g_k the corresponding Riemannian metric. We can assume that $t_k \geq t_0$ for some fixed t_0. We also assume that $T < 1$, since if $T = 1$, the exact same proof of Chen-Donaldson-Sun also works in our case.

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Gromov-Hausdorff convergence

- The metrics satisfy
  1. $\mathrm{Vol}(M, g_k) = V := \frac{(2\pi)^n}{n!} c_1(M)^n$.
  2. $Ric(g_k) \geq t_0 g_k$.
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- (Gromov) The sequence of Riemannian manifolds $(M, g_k) \xrightarrow{G-H} (Z, d)$, where (Z, d) is a compact metric length space.
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Metric geometry to algebraic geometry : Partial C^0-estimate

The following theorem was conjectured by Yau, and made precise, and proved by Tian when $n = 2$.

Theorem (Donaldson-Sun (for KE) [5], Szekelyhidi (in the present context) [7])

There exists a uniform m, and embeddings $F_k : M \hookrightarrow \mathbb{P}^N$ by sections of $H^0(M, K_M^{-m})$ with the following properties

1. *F_k are uniformly Lipschitz.*
2. *$F_k(M)$ converge to a $\mathbb{Q}$-Fano normal flat limit W, and the maps F_k converge to a homeomorphism $F : Z \to W$.*
3. *(partial C^0 estimate) There exists a uniform constant C such that*
$$\omega_k = \frac{1}{m} F_k^* \omega_{FS} + \sqrt{-1}\partial\bar{\partial}\psi_k,$$
with $|\psi_k|_{C^0}, |\nabla\psi_k| < C$.
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- From now on, we denote $F_1(M)$ by M and $F_1(\alpha)$ by α. For simplicity, we assume that $\alpha = m^{-1}\omega_{FS}\Big|_M$.
- Let $\rho_k = F_k \circ F_1^{-1} \in GL^G$, and $M_k = \rho_k(M) \to W$. Also $\rho_k(\alpha) \to \beta$.
- The partial C^0 then says that there exists C such that $|\psi_k|_{C^0} < C$ and

$$\omega_k = \frac{1}{m}\rho_k^*\omega_{FS} + \sqrt{-1}\partial\bar{\partial}\psi_k.$$

Proposition

After passing to a subsequence, $\rho_k \to g \in GL^G$.

Proof of main theorem.

Proposition $\Longrightarrow$

$$\frac{1}{m}\rho_k^*\omega_{FS} - \frac{1}{m}\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\nu_k$$

with $|\nu_k|_{C^0} < C$. Since $\omega_k = m^{-1}\omega_{FS} + \sqrt{-1}\partial\bar{\partial}\varphi_k$, $\varphi_k = \psi_k + \nu_k$, and so combined with partial C^0, $|\varphi_k|_{C^0} < C$.

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Twisted KE on W

- The currents $\rho_k(\omega_k)$ converge weakly to a weak current ω_T supported on W solving the twisted KE equation on W in the weak sense -

$$\mathrm{Ric}(\omega_T) = T\omega_T + (1-T)\beta.$$

- Consequences -
  1. "$\mathrm{Aut}(W, \beta)$" is reductive.
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- W is in the GL^G orbit closure but might not be accessible by $\mathbb{C}^*$.
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Extremal metrics

- Let $L \to M$ be an ample line bundle.
- (Calabi, 1980s) The critical points of the functional

$$Ca(\omega) = \int_M ||\mathrm{Rm}(\omega)||^2,$$

as ω varies over Kähler metrics in the fixed co-homology class $2\pi c_1(L)$ are called extremal metrics.

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where $\mathbf{s}_\omega$ is the scalar curvature.

- In particular constant scalar curvature Kähler metrics (cscK), and hence Kähler-Einstein metrics, are automatically extremal.
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Perturbation problems

Suppose M admits an extremal metric ω. If $p \in M$, and $\pi : Bl_p(M) \to M$ is the blow-up with exceptional divisor E, then it is known that $L_\varepsilon = \pi^*[\omega] - \varepsilon^2[E]$ is Kähler for $\varepsilon << 1$.

Question

If $(Bl_p(M), L_\varepsilon)$ is relatively K-stable, does it admit an extremal metric?

- (Szekelyhidi) If $n > 2$, the answer is affirmative for cscK metrics.
- When $n = 2$, I have recently made some progress [3], but an optimal result is still missing.
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Convergence of cscK manifolds and partial C^0 estimate

- Sequences of cscK manifolds might collapse.

Question

Is there a uniform partial C^0 estimate along a sequence of non-collapsed, cscK metrics on Kähler manifolds with uniform bounds on the total volume, and Calabi energy?

- For $n = 2$, there is an optimal convergence result due to Anderson and Tian-Viaclovsky.
- For $n > 2$, the convergence result assumes $L^{n/2}$ bound on $||\mathrm{Rm}||$, which is not useful, since $Ca(\omega)$ involves an L^2 bound.
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Thank You for your attention!

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