NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, February 2, 2008

Time Allowed: Two Hours

Maximum Marks: 40

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 9 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.

- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.

- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit**.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol $\mathbb{Z}_n$ will denote the ring of integers modulo n. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup' norm. The space of continuously differentiable real valued functions on $[a, b]$ is denoted by $\mathcal{C}^1[a, b]$ and its usual norm is the maximum of the sup-norms of the function and its derivative.

Section 1: Algebra

1.1 Let S_7 denote the group of permutations of 7 symbols. Find the order of the permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 5 & 7 & 3 & 1 & 2 \end{pmatrix}.$$

1.2 Write down the number of mutually nonisomorphic abelian groups of order 19^5.

1.3 For two ideals $\mathcal{I}$ and $\mathcal{J}$ in a commutative ring $\mathcal{R}$, define $\mathcal{I} : \mathcal{J} = \{a \in \mathcal{R} : a\mathcal{J} \subset \mathcal{I}\}$. In the ring $\mathbb{Z}$ of all integers, if $\mathcal{I} = 12\mathbb{Z}$ and $\mathcal{J} = 8\mathbb{Z}$, find $\mathcal{I} : \mathcal{J}$.

1.4 Let $\mathcal{P}$ be a prime ideal in a commutative ring $\mathcal{R}$ and let $S = \mathcal{R} \backslash \mathcal{P}$, *i.e.* the complement of $\mathcal{P}$ in $\mathcal{R}$. Pick out the true statements:
(a) S is closed under addition.
(b) S is closed under multiplication.
(c) S is closed under addition and multiplication.

1.5 Let p be a prime and consider the field $\mathbb{Z}_p$. List the primes for which the following system of linear equations DOES NOT have a solution in $\mathbb{Z}_p$:

$$\begin{aligned} 5x + 3y &= 4 \\ 3x + 6y &= 1. \end{aligned}$$

1.6 Let A be a 227×227 matrix with entries in $\mathbb{Z}_{227}$, such that all its eigenvalues are distinct. Write down its trace.

1.7 Let B be a nilpotent $n \times n$ matrix with complex entries. Set $A = B - I$. Write down the determinant of A.

1.8 Let $\mathbf{x}$ and $\mathbf{y}$ be two non-zero $n \times 1$ vectors. If $\mathbf{y}^T$ denotes the transpose of $\mathbf{y}$, what are the eigenvalues of the $n \times n$ matrix $\mathbf{x}\mathbf{y}^T$?

1.9 Let A be a real symmetric $n \times n$ matrix whose only eigenvalues are 0 and 1. Let the dimension of the null space of $A - I$ be m. Pick out the true statements:
(a) The characteristic polynomial of A is $(\lambda - 1)^m \lambda^{m-n}$.
(b) $A^k = A^{k+1}$ for all positive integers k.
(c) The rank of A is m.

1.10 What is the dimension of the space of all $n \times n$ matrices with real entries which are such that the sum of the entries in the first row and the sum of the diagonal entries are both zero?

Section 2: Analysis

2.1 Let $f(x) = \frac{1}{1+x^2}$. Consider its Taylor expansion about a point $a \in \mathbb{R}$, given by $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$. What is the radius of convergence of this series?

2.2 Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}.$$

Pick out the true statements:
(a) The series converges for all real values of x.
(b) The series converges uniformly on $\mathbb{R}$.
(c) The series does not converge absolutely for any real value of x.

2.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which of the following imply that it is uniformly continuous?
(a) f is 2π-periodic.
(b) f is differentiable and its derivative is bounded on $\mathbb{R}$.
(c) f is absolutely continuous.

2.4 Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Assume that $\int_{-1}^{1} f(t)\ dt = 1$. Evaluate:

$$\lim_{n\to\infty} \int_{-1}^{1} f(t) \cos^2 nt\ dt.$$

2.5 Let f be a continuously differentiable 2π-periodic real valued function on the real line. Let $a_n = \int_{-\pi}^{\pi} f(t) \cos nt\ dt$ where n is a non-negative integer. Pick out the true statements:
(a) The derivative of f is also a 2π-periodic function.
(b) $|a_n| \leq C\frac{1}{n}$ for all n, where $C > 0$ is a constant independent of n.
(c) $a_n \to 0$, as $n \to \infty$.

2.6 Let f_n and f be continuous functions on an interval $[a, b]$ and assume that $f_n \to f$ uniformly on $[a, b]$. Pick out the true statements:
(a) If f_n are all Riemann integrable, then f is Riemann integrable.
(b) If f_n are all continuously differentiable, then f is continuously differentiable.
(c) If $x_n \to x$ in $[a, b]$, then $f_n(x_n) \to f(x)$.

2.7 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $f'(0) = 0$. Define, for x and $y \in \mathbb{R}$,

$$g(x, y) \;=\; f(\sqrt{x^2 + y^2}).$$

Pick out the true statements:
(a) g is a differentiable function on $\mathbb{R}^2$.
(b) g is a differentiable function on $\mathbb{R}^2$ if, and only if, $f(0) = 0$.
(c) g is differentiable only on $\mathbb{R}^2 \backslash \{(0, 0)\}$.

2.8 Find the square roots of the complex number $1 + 2i$.

2.9 Evaluate:

$$\int_{|z|=1} \frac{4 + z}{(2 - z)z} \, dz$$

the circle $\{|z| = 1\}$ being described in the anticlockwise direction.

2.10 Pick out the true statements:
(a) There exists an analytic function f on $\mathbb{C}$ such that $f(2i) = 0, f(0) = 2i$ and $|f(z)| \leq 2$ for all $z \in \mathbb{C}$.
(b) There exists an analytic function f in the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ such that $f(\frac{1}{2}) = 1$ and $f(\frac{1}{2^n}) = 0$ for all integers $n \geq 2$.
(c) There exists an analytic function f on $\mathbb{C}$ whose real part is given by $u(x, y) = x^2 + y^2$, where $z = x + iy$.

Section 3: Topology

3.1 Which of the following define a metric on $\mathbb{R}$?
(a)

$$d_1(x,y) \;=\; \frac{|\,|x|-|y|\,|}{1+|x|.|y|}.$$

(b)

$$d_2(x,y) \;=\; \sqrt{|x-y|}.$$

(c)

$$d_3(x,y) \;=\; |f(x)-f(y)|$$

where $f:\mathbb{R}\to\mathbb{R}$ is a strictly monotonically increasing function.

3.2 Let X and Y be topological spaces and let $f : X \to Y$ be a continuous bijection. Under which of the following conditions will f be a homeomorphism?
(a) X and Y are complete metric spaces.
(b) X and Y are Banach spaces and f is linear.
(c) X is a compact topological space and Y is Hausdorff.

3.3 Let (X,d) be a compact metric space and let $\{f_\alpha : \alpha \in A\}$ be a uniformly bounded and equicontinuous family of functions on X. Define

$$f(x) \;=\; \sup_{\alpha\in A} f_\alpha(x).$$

Pick out the true statements:
(a) For any $t\in\mathbb{R}$, the set $\{x\in X : f(x) < t\}$ is an open set in X.
(b) The function f is continuous.
(c) Every sequence $\{f_{\alpha_n}\}$ contained in the above family admits a uniformly convergent subsequence.

3.4 Let $D = \{x\in\mathbb{R}^2 : |x|\le 1\}$ where $|x|$ is the usual euclidean norm of the vector x. Let $f : D\to X$ be a continuous function into a topological space X. Pick out the cases below when f will NEVER be onto.
(a) $X = [-1,1]$.
(b) $X = [-1,1]\backslash\{0\}$.
(c) $X = \;]-1,1[$.

3.5 Let $\mathbb{M}_n(\mathbb{R})$ denote the set of all $n\times n$ matrices with real entries, considered as the space $\mathbb{R}^{n^2}$. Which of the following subsets are compact?
(a) The set of all invertible matrices.
(b) The set of all orthogonal matrices.
(c) The set of all matrices whose trace is zero.

3.6 With the notations as in the preceding question, which of the following sets are connected?
(a) The set of all invertible matrices.
(b) The set of all orthogonal matrices.
(c) The set of all matrices whose trace is zero.

3.7 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define

$$G \;=\; \{(x, f(x)) : x \in \mathbb{R}\} \;\subset \mathbb{R}^2.$$

Pick out the true statements:
(a) G is closed in $\mathbb{R}^2$.
(b) G is open in $\mathbb{R}^2$.
(c) G is connected in $\mathbb{R}^2$.

3.8 Let X be a topological space and let $A \subset X$. Let ∂A denote the boundary of A, *i.e.* the set of points in the closure of A which are not in the interior of A. A closed set is *nowhere dense* if its interior is the empty set. Pick out the true statements:
(a) If A is open, then ∂A is nowhere dense.
(b) If A is closed, then ∂A is nowhere dense.
(c) If A is any subset, then ∂A is always nowhere dense.

3.9 Let V be a complete normed linear space and let B be a basis for V as a vector space. Pick ot the true statements:
(a) B can be a finite set.
(b) B can be a countably infinite set.
(c) If B is infinite, then it must be an uncountable set.

3.10 Let $V_1 = \mathcal{C}[0, 1]$ with the metric

$$d_1(f, g) \;=\; \max_{t \in [0,1]} |f(t) - g(t)|.$$

Let $V_2 = \mathcal{C}[0, 1]$ with the metric

$$d_2(f, g) \;=\; \int_0^1 |f(t) - g(t)| \; dt.$$

Let id denote the identity map on $\mathcal{C}[0, 1]$. Pick out the true statements:
(a) $id : V_1 \to V_2$ is continuous.
(b) $id : V_2 \to V_1$ is continuous.
(c) $id : V_1 \to V_2$ is a homeomorphism.

Section 4: Applied Mathematics

4.1 Write down the Laplace transform of the function $f(x) = \cos 2x$.

4.2 Let B denote the unit ball in $\mathbb{R}^N$, $N \geq 2$. Let α_N be its (N-dimensional) volume and let β_N be its (($N-1$)-dimensional) surface measure. Apply Gauss' divergence theorem to the vector field $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ and derive the relation connecting α_N and β_N.

4.3 Assume that the rate at which a body cools is proportional to the difference in temperature between the body and its surroundings. A body is heated to 110°C and is placed in air at 10°C. After one hour, its temperature is 60°C. At what time will its temperature reach 30°C?

4.4 A body of mass m falls under gravity and is retarded by a force proportional to its velocity. Write down the differential equation satisfied by the velocity $v(t)$ at time t.

4.5 Solve the above equation given that the velocity is zero at time $t = 0$.

4.6 Write down the critical points of the nonlinear system of differential equations:

$$\begin{aligned} \tfrac{dx}{dt} &= y(x^2+1) \\ \tfrac{dy}{dt} &= 2xy^2. \end{aligned}$$

4.7 Classify the following differential operator as elliptic, hyperbolic or parabolic:

$$\mathcal{L}(u) \;=\; 2\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2}.$$

4.8 Let B be the unit ball in the plane and let u be a solution of the boundary value problem:

$$\begin{aligned} \Delta u &= C \text{ in } B \\ \tfrac{\partial u}{\partial n} &= 1 \text{ on } \partial B \end{aligned}$$

where Δ denotes the Laplace operator, ∂B denotes the boundary of B and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative on the boundary. Evaluate C, given that it is a constant.

4.9 Write down the dual of the linear programming problem:

$$\text{Max. : } 2x + 3y$$

such that

$$\begin{aligned} x + 2y &= 3 \\ 2x + y &\geq 4 \\ x + y &\leq 5 \\ x \geq 0 \quad &, \quad y \geq 0. \end{aligned}$$

4.10 Write down the Newton-Raphson iteration scheme to find the square root of $a > 0$, by solving the equation $x^2 = a$.

Section 5: Miscellaneous

5.1 Differentiate with respect to t:

$$f(t) \ = \ \int_{-t^2}^{a} e^{-x^2}\ dx, \ (a > 0).$$

5.2 Sum the series:

$$1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \cdots$$

5.3 Sum the series:

$$1 + \frac{1}{3}\,\frac{1}{4} + \frac{1}{5}\,\frac{1}{4^2} + \frac{1}{7}\,\frac{1}{4^3} + \cdots$$

5.4 Let A be the point $(0,1)$ and B the point $(2,2)$ in the plane. Consider all paths made up of the two line segments AC and CB as the point C varies on the x-axis. Find the coordinates of C for which the corresponding path has the shortest length.

5.5 Find the area of the pentagon formed in the plane with the fifth roots of unity as its vertices.

5.6 Let A, B, C, D and E be five points marked, in clockwise order, on the unit circle in the plane (with centre at origin). Let α and β be real numbers and set $f(P) = \alpha x + \beta y$ where P is a point whose coordinates are (x, y). Assume that $f(A) = 10, f(B) = 5, f(C) = 4$ and $f(D) = 10$. Which of the following are impossible?
(a) $f(E) = 2$
(b) $f(E) = 4$
(c) $f(E) = 5$

5.7 Let r identical red balls and b identical black balls be arranged in a row. Write down the number of arrangements for which the last ball is black.

5.8 It is known that a family has two children.
(a) If it is known that one of the children is a girl, what is the probability that the other child is also a girl?
(b) If it is known that the elder child is a girl, what is the probability that the younger child is also a girl?

5.9 A real number is called *algebraic* if it occurs as the root of a polynomial with integer coefficients. Otherwise it is said to be a *transcendental* number. Consider the interval $[0, 1]$ considered as a probability space when it is provided with the Lebesgue measure. What is the probability that a number chosen at random in $[0, 1]$ is transcendental?

5.10 List all primes $p \leq 13$ such that p divides $n^{13} - n$ for every integer n.