Cones, Positive Operators and their spectral properties in real Banach spaces

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Starting with Statistical Decision Functions

In chapter 2, Wald starts with zero sum two person games. Here mother Nature tries

to hide the true value of a parameter, though the statistician is allowed

to have a partial peek of the parameter through a random sample $x = (x_1, x_2, \ldots, x_n)$ of

fixed size n. Statistician has to take a decision $d = d(x)$ as a function of the data x collected

, resulting in a bounded risk $r(\theta, \quad d)$. Wald took the bold view that the Statistical Decision

problem is simply a zero sum two person game between Nature and the statistician. Mother

Nature pegs its choice on a prior probability distribution ξ on the parameter space Θ and

independently statistician pegs his/her choice on a probability distribution η over the space

D of all decision functions. The game is played according to these resulting in a the bilinear

functional

Zero sum two person Games

$$\int_{\Theta\times D} r(\theta, d) \quad d\xi d\eta.$$

Wald goes through various ways to topologize the two spaces Θ and D using the bounded function $r(\theta, \quad d)$. As one just starting to learn set topology from Professor SRS Varadhan's lectures, I was completely lost in Wald's inteplay with weak topolgy and so on. It was more

to do with zero sum two person games on infinite spaces

Von Neumann-Kaplanski-Blackwell

While one can find the unique value v of an $n \times n$ matrix A with strictly positive entries, (say by the Simplex method) it was not clear as to how this can help to find the unique value of the spectral radius and the other assertions of the celebrated Perron-Frobenius Theorem.

These theorems of von Neumann, Kaplanski and Blackwell can be summarized as follows:

Minimx Theorem (von Neumann) Let $A = (a_{ij})$ be an $n \times n$ matrix. Then there exists a unique constant v and a pair of probability vectors $x = (x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ such that

$$\sum_{i=1}^{n} a_{ij} x_i \geq v \quad \forall \quad j = 1, \ldots, n.$$

$$\sum_{i=1}^{n} a_{ij} y_j \leq v \quad \forall \quad i = 1, \ldots, n.$$

Completely mixed games

Completely mixed games: (Kaplanski) Let A be an $n \times n$ payoff matrix with $v = 0$. If every optimal strategy (probability vector) for one player is strictly positive, then so it is for the other player and

1. The optimal probability vectors x, y are unique for the two players
2. $Ay = \mathbf{0}, \quad A^t x = \mathbf{0}$
3. Rank of A is $n - 1$
4. All cofactors of A are different from zero and are of the same sign.

PF Theorem via Minimax Theorem

Theorem:(Blackwell) Let A be an $n \times n$ matrix with all entries positive. Let $v(\lambda)$ be the value of the matrix game $(A - \lambda I)$. Then

1. $v(\lambda) \to \pm\infty$ when $\lambda \to (-) \pm \infty$.
2. $v(\lambda)$ is Lipschitz continuous.
3. $v(\lambda)$ is strictly decreasing in a neighborhood of λ_0 when $v(\lambda_0) = 0$.
4. The game $A - \lambda_0 I$ is completely mixed.
5. λ_0 is an algebraically and hence a geometrically simple root of the characteristic equation.
6. λ_0 is the spectral radius.

Spectrum, regular points and spectral radius of bounded linear oprators

We will always assume E as a Banach space over the real field.

Let $||.||$ denote the norm and for any bounded linear operator A we denote by $||A||$ the $\sup_{||x||\leq 1} ||Ax||$.

The spectral radius ρ of a linear operator A is defined by $\rho = \lim_n \sqrt[n]{||A^n||}$. We can extend E to $\tilde{E}$ where for any $x, y \in E$ we define $z = x + iy \in \tilde{E}$ and for any complex number $\zeta = \alpha + i\beta$ we define $A\zeta x = \alpha Ax + i\beta Ax$. Here $||z|| = \max_{0\leq\theta\leq\pi} ||xcos\theta + ysin\theta||$

Definition: We call any $\lambda \in \mathcal{C}$ regular if $(A - \lambda I) : \tilde{E} \longrightarrow \tilde{E}$: is one-one onto and bicontinuous. The spectrum $\sigma(A)$ is the complement of regular points .

Remark: Unlike in finite dimensions, there may be no characteristic vector for elements in $\sigma(A)$. A typical example will be the multiplication operator $A : E \longrightarrow f(t) \to tf(t)$ in $C[0 \quad 1]$.

Cones and linear semi groups in real Banach spaces

E a real Banach space

$K \subset E$ is called a linear semi group iff for any $\lambda, \mu \geq 0 \quad x, y \in K \implies \lambda x + \mu y \in K$.

A linear semi group K is called a cone if $x \in K - \theta \implies -x \notin K$.

Example 1: $E = C[0 \quad 1]$. For any $f \in E$ we define $||f|| = \max_{0 \leq t \leq 1} |f(t)|$. Let K be all polynomials with real coefficients. Here K is a linear semigroup.

Example 2: $E = C[0 \quad 1]$. Let K be all polynomials with nonnegative coefficients. Here K is a cone.

Example 3: $E = C[0 \quad 1] \cap \{f : f(0) \geq 0\}$. Let $K =$ all polynomials of the type $\sum_{i=0}^{n} \alpha_k (-t)^k, \quad \alpha_k \geq 0 \quad k = 0 \ldots, n; \quad n = 1, 2 \ldots$

On cones with interior

Given a cone $K \subset E$ we call $A : E \longrightarrow E$ a positive operator iff $AK \subseteq K$. The cone induces a partial order $\geq$ on some elements of E. We say $x \geq y$ iff $x - y \in K$. Let E^* be the set of all bounded linear functionals on E. We denote by $K^* = \{\psi : \psi \in E^*, \quad \psi(x) \geq 0 \quad \forall x \in K\}$. We call K a *reproducing* cone if $E = K - K$. When the interior of K exists, we denote it by K^0.

Theorem: Given a cone $K \subset$ with non-null interior, any additive functional ψ satisfying $\psi(x) \geq 0 \quad \forall x \in K$ is in K^*. That is ψ is a continuous linear functional on E.

Theorem: Let K be a cone with interior and let $\psi \in K^*$. If $u \in K^0$ with a sphere of radius ρ and center u contained in K we have $\psi(u) \geq \rho||\psi||$

Normal cones

Normal cones

Definition A cone K in E is called a *Normal cone* iff there exists a $\delta >$ such that for any two elements $x, y \in K$ with $||x|| = ||y|| = 1 \quad ||x + y|| \geq \delta > 0$ where δ is independent of $x, y \in K$. Another equivalent definition is that there exists a $\delta > 0$ such that for any two arbitrary elements $x, y \in K, \quad ||x + y|| \geq \delta\{\max ||x||, ||y||\}$.

Theorem: Let $u \in K^0$ for a cone K. The cone K is normal if and only if the set

$$I_u = \{-u \leq x \leq u\} \text{ is bounded.}$$

Commuting positive Operators with a common eigenvector

Theorem: Let $\{A_\gamma\} \quad \gamma \in \Gamma$ be a commuting family of bounded linear operators on E and let $A_\gamma K^0 \subset K^0 \quad \forall \gamma \in \Gamma$. Then for some common $\psi \in K^* - \theta$ we have $A_\gamma^* \psi = \lambda_\gamma \psi$ with $\lambda_\gamma > 0, \gamma \in \Gamma$. Thus the dual operators all have a common characteristic vector.

Proof: Let $u \in K^0$ and fix one of the operators say $A \in \Gamma$, and let $v = Au >> \theta$. Let $S(u, \quad \rho) \subset K \implies u \pm \rho e \in K$ when $|e| = 1$. For any $\phi \in K^*$ we have $\phi(u) \pm \rho\phi(e) \geq 0$.
When $v = Au$ we have $\phi(v) = \phi(Au) = (A^*\phi)(u) \geq \sigma||\phi||$ for some $\sigma > 0$.
Consider the compact convex set

$$H = \{\phi : \phi \in K^*, \quad \phi(u) = \rho\}$$

Proof via Markov-Tychonoff Fixed Point Theorem

H is weakly compact and the map

$$B\phi = \rho \frac{A^*\phi}{\phi(v)}$$

is a weakly continuous self map of H into itself.

$B\psi = \psi$

$H_1 = \{\psi : A_1^*\psi = \lambda_1\psi\}.$

compactness of $\cap_j H_j$

$A_\gamma^*\psi_0 = \lambda_\gamma\psi_0$

Why for the dual and not for the original

Example: Let $E = C[0 \quad 1]$. Let $A : E \longrightarrow E$ where A sends $f(t) \to tf(t)$. Since $0 \leq t \leq 1$ given any $g \in E$ we have a solution f to $(A - \lambda I)f = g$ only when $\lambda \notin [0 \quad 1]$. Thus $\sigma(A) = [0 \quad 1]$. The spectral radius of A is 1. If the spectral radius has to possess a characteristic vector, then we should have $(A - I)f = \theta$ namely $f(t) = tf(t) \quad \forall \quad 0 \leq t \leq 1$. The only solution is $f = \theta$. Since the cone K of nonnegative functions in $C[0 \quad 1]$ is a normal cone with interior, we only claim that the spectral radius is a characteristic value for the dual opertaor A^*. Since by Riesz representation theorem, any $\psi \in E^*$ is a signed measure on $[0 \quad 1]$ all we demand is to look for a solution to the equation

$$\int_0^1 tf(t)d\psi(t) = \int_0^1 f(t)d\psi(t) \quad \forall f \in E.$$

Clearly the dirac measure $\psi = \delta_1$ is the required candidate.

Normal cones and spectral radius

Theorem: Let K be a normal cone with interior in E and let $AK \subset K$. Suppose $Au = \rho u$ for some $u \in K^0$. Then ρ is the spectral radius.

Remark: Assumption that $K^0 \neq \emptyset$ is critical. For example let $E = l^2$ and $K = \{x : x = \{\xi_n\} : \xi_n \geq 0 \quad n = 1, 2, ..\} \cap l^2$

Let $A : \{\xi_n\} \to \{0, \{\xi_n\}\}$ We observe that the cone has empty interior. Even though the operator A is linear and maps K into itslf, since $K = K^*$ if $A^*\psi = \lambda\psi$ this implies $(A^*\psi \quad x) = (\psi \quad Ax) \quad \forall x \in l^2$. We observe $||Ax|| = ||x||$ and so the spectral radius of A is 1. We can write $\psi(x) = \sum_j \psi_j x_j$. Suppose $A^*\psi = \psi$. We get $(\psi_2 x_1 + \psi_3 x_2 + \ldots) = (\psi_1 x_1 + \psi_2 x_2 + \psi_3 x_3 + \ldots) \quad \forall x \in K$. If we take $x = e_1$, the unit vector, we get $\psi_2 = \psi_1$. If we take $x = e_2$ we get $\psi_3 = \psi_2$ and so on. Such a ψ vector cannot be an element of l^2 except for $\psi = \theta$. Thus we see that the spectral radius is not a characteristic number of A^*.

Positive operators in reflexive Banach spaces via the minimax theorem of Ky Fan

Theorem: Let E be a reflexive Banach space and $K \subset E$ be a closed cone with interior and let the conjugate cone K^* also be one with non-null interior. Let $AK^0 \subset K^0$ and let $Ax \neq \theta$ if $x \in K - \theta$. Then there exists a vector $u \in K - \theta$ and $\psi \in K^* - \theta$ such that

$$Au = \rho u, \quad A^*\psi = \rho\psi \quad (\rho > 0).$$

The spectrum $\sigma(A) \subset \{\lambda : |\lambda| \leq \rho\}$

Let K_1, K_2 be compact convex sets in locally convex linear topological spaces E_1 and E_2 respectively

Let $L(x, y)$ be a bileanear functional on $K_1 \times K_2$ such that it is continuous in each variable

$$\max_{x\in K_1} \min_{y\in K_2} L(x,y) = \min_{y\in K_2} \max_{x\in K_1} L(x,y) = L(x^0, y^0)$$

Positive Operators on reflexive Banach spaces

Theorem: Let K be a closed cone in a reflexive Banach space E. Let K, K^* have non-empty interior. Further let A be a strongly positive bounded linear operator. Then

1. The spectral radius λ_0 is an eigen value of A.
2. There exists an eigen vector z for λ_0 with $z > \theta$.
3. The subspace $S_{\lambda_0} = \{y : Ay = \lambda_0 y\}$ is one dimensional.
4. A^* has an eigenvector ϕ for λ_0 which is strictly positive on $K^* - \theta$.
5. No other linearly independent eigenvector of A or A^* lie in K or K^*.

Normal cones and cones with interior play a critical role

Spectral radius is outside the spectrum

Example: Let $E = C[0 \quad 1]$. Let K= all polynomials $\sum_{i=0}^{n} \alpha_k(-t)^k, \quad \alpha_k \geq 0 \quad k = 0 \ldots, n; \quad n = 1, 2 \ldots$ generate a dense linear manifold in $C[0 \quad 1]$. The operator $A : f(t) \to -tf(t)$ maps the cone K into itself. The spectrum is $[-1 \quad 0]$ while the spectral radius is 1.

A querry: If the cone K has interior can we say that the spectral radius of any positive operator is in the spectrum? Not necessarily.

Example: Let E : be all complex valued functions over the real field R and continuous on the unit disk, real valued on $[-1, \quad 1]$ and regular on $\{\zeta : |\zeta| < 1\}$. One observes that E is a Banach space. Let K b the cone of nonnegative functions continuous on $[-1 \quad -\frac{1}{2}]$. The function $u(\zeta) = 1 \quad \forall \quad |\zeta| \leq 1$ is interior to the cone.

$$A : f(\zeta) = -(\zeta + \tfrac{1}{2})f(\zeta)$$

Cone with nonnull interior alone will not do

spectrum of A intersects the real axis $[-1 \quad \frac{1}{2}]$

no positive linear functional as the characteristic vector for the spectral radius

for A^* even though the cone has interior points.

Even though the spectral radius is not in the spectrum, Krein's theorem guarantees $A^*\psi = \lambda\psi, \quad \lambda > 0$ and $\psi \in K^*$ is valid and we have in fact $A^*\psi = \frac{1}{2}\psi$ for $\psi(f) = f(-1)$. Also $\psi \in K^*$.

Problems with continuity of operators

Remark: A linear operator may be continuous on a cone K, but may fail to be continuous on E even though the cone generates a dense linear manifold in E. Here is one such example.

Let $E = C[0 \quad 1] \cap \{f(0) = 0\} \quad K = \{f : f \geq 0, \text{and convex}\}$

Consider the operator $A : f(t) \to f(\frac{t}{2}) \quad 0 \leq t \leq 1$

Observe that for any $f \in K \quad [((I - A)^{-1})f](t) = f(t) + f(\frac{t}{2}) + f(\frac{t}{4}) + \ldots$ is well defined for $0 \leq t \leq 1$. Reason: $0 \leq f(t) \leq f(1)$ and convexity shows that the series is uniformly bounded by $2f(1)$. However it is not so for $f \in K - K$. Consider $f_n(t) = g_n(t) - h_n(t)$ where

$$
\begin{aligned}
g_n(t) &= 2^n t & 0 \leq t \leq 1 \\
h_n(t) &= 0 & 0 \leq t \leq 2^{-n} \\
&= 2^n t - 1 & 2^{-n} \leq t \leq 1
\end{aligned}
$$

Observe that $\|f_n\| = 1$ while $\sum^{\infty} f_n(2^{-k}) \geq \sum^{n} f_n(2^{-k}) = n + 1$

Partial spectral radius

Remark: The above operator A has the following additional properties

1. A is compact in K
2. A is not compact in E
3. (iii) If S is the unit sphere in E then if $p(A) = \lim_n \sup_{x \in K \cap S} \sqrt[n]{A^n x}$ then $p(A) = \frac{1}{2}$ and $Au = \frac{1}{2}u$ for some $u \in K - \theta$.

Lorentz Bilinear Functional

Let H be a real Hilbert space with the unit vectors $e_n, \quad n = 1, 2, \ldots$. Let J be an operator defined by $Jx = x_1 e_1 - \sum_{j=2}^{\infty} x_j e_j$. We have $J^2 = I$ Let us denote the bilinear functional

$(Jx \quad y) =< x \quad y >$.

Let $K = \{x : (xe_1) \geq 0, < x \quad x > \geq 0\}$

Theorem: If $x, y \in K$ then $< x \quad y > \quad \geq \sqrt{< x \quad x >}\sqrt{< y \quad y >}$

Definition: We call a map $A : H \to H$ a Lorentz map if A is one:one such that

$$< Ax, \quad Ay >= \quad < x, \quad y > \forall x, y \in H$$

Lorentz Operators on Lorentz cone

Theorem: If $\lambda \in \sigma(A)$ then $\frac{1}{\lambda}$ is in $\sigma(A)$. Also if $Ax = \lambda x, \quad Ay = \mu y$ and if $\lambda\bar{\mu} \neq 1$ then $< x, \quad y > \quad = 0$.

Theorem: If $x \in K$ then Ax or $-Ax \in K$. Also if for one $x_0 \neq 0$, if x_0 and Ax_0 are both in K then $AK \subset K$. In fact $AK = K$.

Theorem: Let A be a Lorentz mapping with $AK \subset K$. Then A has a characteristic vector $v \in K$ such that

i. $\rho = 1$ and the whole spectrum of A lies on the unit circle.

ii. $\rho > 1$ In this case A has another characteristic vector $u \in K$ such that $Au = \frac{1}{\rho}u$.

In this second case both v and u belong to the boundary of k.

Compact Operators in Banach spaces

Definition: A bounded linear operator $A : E \to E$ is a compact operator iff the range of every bounded set is a conditionally compact set in E.

$$A : \phi \to \int_0^1 K(s,t)\phi(t)dt$$

1. 1. $K(s,t)$ is measurable and integrable with respect to t.

2. 2. $\lim_{h\to 0} \int_0^1 |K(s+h,t) - K(s,t)|dt = 0.$

maps bounded sets of $C[0\quad 1]$ to conditionally compact sets via Ascoli-Arzela theorem (equicontiuity property).

Properties of compact operators in real or complex Banach spaces.

1. The spectrum $\sigma(A)$ of any compact operator in Banach spaces of infinite dimensions will have a countable number of characteristic values and with 0 as the only limit point.

2. $\sigma(A) \subseteq \{\lambda : |\lambda| \leq \rho\} = \lim_{n\to\infty} \sqrt[n]{||A^n||}$

3. In a suitable neighbor hood $N(\lambda)$ of any nonzero eigenvalue λ_0 the resolvent operator

$$R_\lambda = (A - \lambda I)^{-1} = \sum_{k=-n}^{\infty} \Gamma_k (\lambda - \lambda_0)^k$$

Here Γ_k , for $k < 0$ have finite dimensional range as bounded linear operators.

Properties of Compact Operators

4. For each $\lambda_0 \neq 0$ in the spectrum we have a positive integer p with a p dimensional null space G_p for $(A - \lambda_0 I)$. We call p the rank when no $q < p$ has this property. We call λ_0 simple when $p = 1$. In this case we say λ_0 is a simple pole of R_λ.

5. A and A^* have the same rank for each $\lambda \in$ the spectrum.

6. λ_0 is a simple eigen value iff the eigenvactor $Av = \lambda_0 v, \quad A^*\psi = \lambda_0\psi$ implies $\psi(v) \neq 0$.

7. If $\lambda \in$ the spectrum of A , then for any given $y \in E$, the equation $Ax - \lambda x = y$ has a solution iff y is orthogonal to any eigen vector $\psi \in E^*$ for A^*.

Compact Positive Operators:

Theorem: Let K be a closed cone with $K - K$ dense in a real Banach space E. Let $A : E \to E$ be a compact operator that leaves the cone K invariant ($AK \subseteq K$). Let $\rho > 0$ be the spectral radius of A. Then there exists a $v \in K, v \neq \theta$ such that $Av = \rho v$. Also we have a $\psi \in K^*$ with $A^*\psi = \rho\psi, \psi \neq \theta$.

A querry: Can we replace K by any closed convex set? Not really.

$K = \{(x,y)^t : x \geq 0, y = 1\}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

spectral radius which is 1, has no eigenvector in the convex set.

Operator leaving a convex set invariant

Theorem: Let K be a closed convex set with θ as an extreme point. Let K contain spheres of arbitrary radius. and let the compact operator A with a spectral radius $\rho > 1$ leave the convex set invariant. Then $Av = \rho v$ for some non-null $v \in K$. Also we have for some $\psi \in E^*, A^*\psi = \rho\psi$ and $\psi(x) \geq 0$ for all $x \in K, \psi \neq \theta$.

A querry: Can we replace $\rho > 1$ by any $\rho > 0$. Not really.

$$K = \{(x, y) : x \geq y^2, x, y \neq 0\}$$

$A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ The spectral radius is $\frac{1}{2}$ and the eigen vector $(0, 1)$ is not in K.

Strongly positive Operators

Let K be a cone with interior. A linear operator A is called *strongly positive* if for every non-null boundary point x of K we have a suitable power $n(x)$ such that $A^{n(x)}x >> \theta$.

Theorem: Let K be a cone with interior in E Let A be a compact strongly positive operator. Then

(a) The spectral radius $\rho > 0$ and $Av = \rho v$ for a unique $v \in K^0; ||v|| = 1$. Further any eigenvector in K^0 is just a scalar multiple of v.

(b) The conjugate operator A^* has one and only one eigenvector ψ in K^* (modulo scalar multiple). We have $A^*\psi = \rho\psi$ where $\psi(x) > 0$ when $x \in K - \theta$

(c) Conversely any compact positive operator is strongly positive when it satisfies (a) and (b).

Eigenvector solution to Mate competition game

Krishna chooses a random time to reach Udupi Hotel sometime between 6PM and 7PM. He will wait for Bhama or Rukmini to show up and will wait for either one till 7PM. Whom soever between the two shows up first after his arrival, he will go with her. The successful mate wins 1 unit from the opponent. Let $t = 0$ at 6PM and $t = 1$ at 7PM. Bhama chooses to show up at $t = x$ and Rukmini chooses to show up at $t = y$ where $0 \leq x, y \leq 1$. Expected payoff to Bhama from Rukmini is given by

$$\begin{aligned} K(x,y) &= x - (y - x) = 2x - y & x < y \\ &= 0 & x = y \\ &= -y + (x - y) = x - 2y & x > y \end{aligned}$$

An intuitive probability density we are looking for

$$\int_a^1 K(x,y)f(x)dx \equiv 0 \quad \forall y \in \text{the spectrum of} \quad f$$

Integration by parts gives the integral equation $\displaystyle\int_a^1 L(x,y)f(x)dx = f(y)$

Here

$$L(x,y) = \frac{1}{y} \quad \text{when} \quad x < y$$

$$= 0 \quad \text{when} \quad x = y$$

$$= \frac{2}{y} \quad \text{when} \quad x > y$$

Induced Compact Positive Operator

$$T_a : g :\longrightarrow \int_a^1 L(x,y)g(x)dx$$

T_a is a compact operator on $C[a \quad 1]$. Since $L(x,y) \geq 0$ for all $a \leq x, y \leq 1$

T_a maps also the cone of nonnegative functions on $[a \quad 1]$ to nonnegative functions. we are looking for a nonnegative solution f_a with $T_a f_a = f_a$

$\int_a^1 f_a(x)dx = 1.$

theorem of Krein and Rutman will guarantee a nonnegative eigenfunction for the spectral radius

Unique density optimal for the two mates

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{4} \\ \frac{1}{2}x^{-\frac{3}{2}} & \frac{1}{4} \le x \le 1 \end{cases}$$

Compact operators in Hilbert spaces

In the Hilbert space E of square summable sequences, given a nonnegative matrix

$[a_{ik}]_1^\infty$ with $$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}|a_{ik}|^2 < \infty.$$

$$y_i = \sum_{k=1}^{\infty} a_{ik}x_k \quad (i = 1, 2, \ldots).$$

This operator maps the minihedral normal cone K of nonnegative sequences into K.

Theorem: If there exists indices $p_1, p_2, \ldots p_m$ such that

$$a_{p_1 p_2}.a_{p_2 p_3}.\ldots.a_{p_{(m-1)} p_m}.a_{p_m p_1} > 0$$

then we have a positive eigenvalue λ_0 with an eigenvector $v \in K$.

Definition: An operator $U : \tilde{E} \to \tilde{E}$ is called a *permutator* iff

1. $D = U\tilde{E}$ is finite dimensional

2. We have a basis $e_1, e_2, \dots e_s$ such that $Ue_j = e_{k_j} \quad (j = 1, 2, \dots s)$,

where

$$T = \begin{pmatrix} 1 & 2 & .. & s \\ k_1 & k_2 & .. & k_s \end{pmatrix}$$

is a permutation of the integers $1, 2 \dots, s$.

If T decomposes into independent cycles $(i_1, i_2, \dots i_\alpha), (j_1, j_2, \dots j_\beta), \dots (m_1, m_2, m_\tau)$, then every eigenvalue of U is a solution of the equation

$$(\zeta^\alpha - 1)(\zeta^\beta - 1)\dots(\zeta^\tau - 1) = 0.$$

Norm functional

Definition: A norm functional $\mu(x) \quad (x \in E)$ is one that satisfies

1. $\mu(x) > 0 \quad$ if $x \neq \theta$;
2. $\mu(\lambda x) = |\lambda| \mu(x)$;
3. $\mu(x + y) \leq \mu(x) + \mu(y)$.

Theorem: Let K be a minihedral cone and let A be a compact positive operator. If for some norm functional $\mu(x)$, $\;, \mu(Ax) \leq \mu(x) \quad (x \in E)$, then the operator A admits a decomposition into orthogonal parts ,

$$A = U_1 + A_1 \quad (A_1 U_1 = U_1 A_1 = \theta), \qquad \sqrt[n]{||A_1^n||} < 1,$$

U_1 is a permutator whose basis lies in K.

Continued

Thus the set of eigenvalues of modulus unity coincides with the set of roots of

$$(\zeta^\alpha - 1)(\zeta^\beta - 1)\dots(\zeta^\tau - 1) = 0.$$

1. Fixed vectors for A and A^* have bases $v_1, v_2, \dots v_r,$ and $\psi_1, \psi_2, \dots \psi_r$ and they lie in the respective cones K and K^*. Further
2. the systems are biorthogonal namely $\psi(v_j) = \delta_{ij} \quad (i, j = 1, 2 \dots, r),$
3. For each $i \neq j, \quad \inf(\psi_i, \psi_j) = \theta$