CHERN AND TOTAL CURVATURE

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Shiing-Shen Chern was one of the great mathematicians of the twentieth century, and a towering figure in the field of Differential Geometry, where one studies *curvatures* of spaces. In this article we give the reader a glimpse into Differential geometry and the work Chern.

1. Wind-vanes on spheres

Let us begin with a few thought experiments. Consider a frictionless wind-vane in vacuum, as in the figure. Assume that the wind-vane is initially stationary. Then if rotate the base of the wind-vane, the arrow does not move (as there is no friction).

Now assume that the wind-vane is placed flat on a plane. Let us move the base of the wind-vane smoothly along a curve in the plane. Then the wind-vane points in a constant direction, i.e., the direction of the wind-vane gives *parallel* vectors along the curve. This gives a physical meaning to vectors at different points being parallel.

Something more interesting happens if we perform the same experiment on the surface of a sphere. Suppose the wind-vane is initially placed at the point A which we take to be the north pole, and points in the direction of B on the equator. Let C be another point on the equator. We shall move the wind-vane along the longitude AB, then along the equator from B to C and finally back to A along a longitude.

As we move along AB, the wind-vane is always pointing in the direction of motion. We can see this by symmetry - as the two sides of the longitude of a sphere look the same, the there is no reason for the wind-vane to turn right rather than left (or vice versa). When we reach B, the wind-vane is perpendicular to the equator, so that it is pointing due south. Now as we move along the equator to C (by symmetry) it continues to point south. Hence when we reach C, we are pointing along the longitude through C towards the south pole. Finally, as we move the wind-vane back along the longitude AC, it continues to point towards C.

Notice that something remarkable has happened. The wind-vane which was pointing towards B is now pointing towards C. Thus if we look at *parallel* vectors

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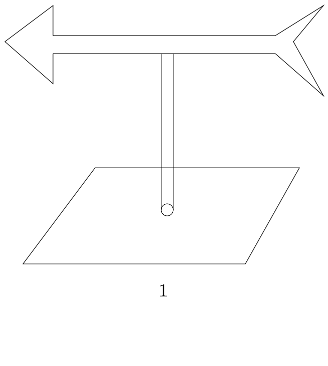


along the loop ABC, we end up with a different vector from the one we had started with. Thus *parallel transport* around a loop on a sphere leads to a rotation, in technical language the *holonomy* of the parallel transport. This happens because the sphere is curved, and the amount by which we rotate depends on the *curvature*. By contrast, in the plane the holonomy is always the identity, i.e., each vector is taken to itself.

2. Parallel transport and curvature

We shall now take a more geometric look at *parallel transport* and *curvature*. Suppose we have a curve α on a surface S from a point P to a point Q and a vector V at P that is tangent to S. Then we can *parallel transport* the vector V along α to get a parallel vector field along α. This corresponds to moving our wind-vane.

Let us see what properties such a parallel vector field should satisfy. Firstly, the vectors of the vector field have the same length. Further, the angle between two parallel vector fields is constant. Properties similar to this will hold for parallel transport in a more general setting. These properties amount to saying that the holonomy about a loop is a rotation.

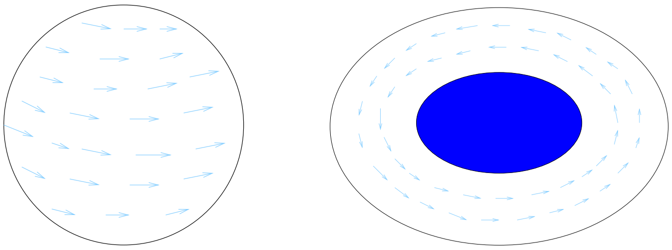
The third property is more subtle. If we move in a straight line in the plane at constant speed, then the *velocity vector* clearly gives a parallel vector field along the line. We have a similar property for more general surfaces. The *shortest* curve on a surface S joining two points P and Q is a *geodesic*. If we move along a geodesic at constant speed, then the velocity vectors are parallel. This condition is called *torsion freeness* and will play a different role from the first two conditions in the general setting.

It turns out that these conditions essentially determine parallel transport. Observe that both these conditions are *intrinsic*, i.e. are determined by distances and angles on the surface and make no reference to the embedding of the surface in Euclidean space. This has an important consequence - we cannot make a map on a plane of a piece of a sphere without distorting distances. This is because if we had such a map, then parallel vector fields on the sphere will map to parallel vector fields on the plane. So the holonomy should be the same in both cases. But we have seen that the holonomy is non-trivial for loops on the sphere, while it is the identity for loops on the plane.

The *curvature* of a surface measures the holonomy. More precisely, let us take a small closed curve, say a parallelogram beginning and ending at a point P. We measure the angle by which parallel transport along the curve rotates vectors. We divide this by the area enclosed by the curve. As we take smaller and smaller curves, this ratio approaches a number which we call the *curvature* $\kappa(P)$ at P.

3. Total Curvature of a sphere

We have seen that we cannot make a map of a piece of the sphere without distorting distances as its curvature is not zero. Suppose we smoothly deform the sphere. We can certainly make part of the sphere flat (i.e., make the curvature in this region zero). But intuition tells us we cannot make the curvature zero everywhere, we can make a region flat only at the cost of making some other region curved.



Indeed our intuition is right, and in fact there is a remarkable result,the Gauss-Bonnet theorem, which tells us that as we deform the sphere, the *total curvature* is unchanged! More precisely, we always have

$$\frac{1}{2\pi}\int_{S^2}\kappa = 2$$

The Gauss-Bonnet theorem can be interpreted as relating the *local* geometric properties of the sphere to its *global* topological properties. Before going into this further, we make a digression into *topology*.

4. The Euler characteristic

Water flowing on a sphere must be stationary at some point. This is not so for water flowing on a torus - the surface of a doughnut (or rubber tube). Formally, any smooth vector field on a sphere (the velocity of the water) has to be zero somewhere, but this is not so for a torus.

The Poincaré-Hopf theorem says this and more. Suppose we have a smooth vector field on a surface S with finitely many zeroes. Then we can associate an index to each zero (similar to the multiplicity of a root of a polynomial). If the surface S is a sphere, then the *total index* is always 2 (remember the 2 in the Gauss-Bonnet theorem)! Similarly, in the case of a torus it is always zero. In general, for a surface S, the total index is always a fixed integer called *the Euler characteristic* $\chi(S)$ of the surface S.

This is a theorem in *topology* because it does not depend on the sphere being round - it is also true for the surface of an egg. Topological properties are those that are preserved by any smooth deformation (or any continuous deformation). The Euler characteristic is such a property - smoothly deforming S does not change $\chi(S)$.

Notice that it is easy to construct a vector field on a piece of a sphere that has no zeroes. In other words, we can make sure that water is not stationary anywhere in a small region. So the Poincaré-Hopf theorem is a *global* property of the sphere. Topological properties are the *global* properties of a space.

In higher dimensions, there are other numbers similar to the Euler characteristic associated to a space. The Euler characteristic is in fact the simplest of the *Chern numbers*, constructed by Chern building on work of Steifel, Whitney and Pontrjagin.

5. More on total curvature

Armed with the Euler characteristic, we now state the more general Gauss-Bonnet theorem. For a surface S, we have

$$\frac{1}{2\pi}\int \kappa = \chi(S)$$

Observe that the integrand on the left hand side is *local* (can be measured on a small piece of the surface), *geometric* (depends on lengths and angles) and *real valued.* However on integrating we get the right hand side, which is *global*, *topological* and an *integer*!

Let us now turn to the higher dimensional situation. Here the notion of curvature is more complicated. Firstly the curvature is different in different directions. More precisely, if we take a small parallelogram near a point P, then the holonomy depends on the plane spanned by the parallelogram.

The holonomy about a loop still preserves lengths and angles, so is still an orthogonal transformation. In 2-dimensions where the orthogonal transformations are just rotations, and so can be specified by a single number which is the angle by which we rotate. This is no longer true in general. For instance, in three dimensions, to specify a rotation we need to also specify its axis.

Nevertheless all these curvatures can be packaged together appropriately to form the *curvature tensor* (in the original approach due to Riemann) or the *curvature form* (in the approach of Elie Cartan). From the curvature form we can extract a real number - the Pfaffian, at each point of our space. The generalised Gauss-Bonnet, due to Allendoerfer, Weil and Chern, says that, in even dimensions, the integral of the Pfaffian is the Euler characteristic.

More generally, the integrals of various other numbers, so called *Chern-Weil integrands*, extracted from the curvature form give various *Chern numbers.* This constitutes *Chern-Weil theory*, whose proper setting is the *differential geometry of fibre spaces*, to which we now turn. This was developed to a large extent by Chern. Chern himself regarded the differential geometry of fibre spaces and his version of the Gauss-Bonnet as two of his most important pieces of work.

6. Bundles, Connections and Curvatures

The differential geometry of fibre spaces comes about by placing parallel transport in its proper setting. The case we consider involved the parallel transport of *tangent vectors* of a surface around a loop, which satisfied two kinds of properties. The first kind of properties ensured that the holonomy was an orthogonal transformation. We also had the condition of *torsion freeness.* We shall generalise all this.

Suppose S is a smooth surface contained in the usual 3-dimensional Euclidean space. At each point P in S, we can consider the tangent space to S at P, which is a vector space. The totality of these tangent spaces constitutes the *tangent bundle.*

There are other ways of associating vector spaces to each point of a space. For instance, for each P in S we can consider the space of vectors orthogonal to S at P. This is a one-dimensional space, and the collection of all such vector spaces forms the *normal bundle.*

The tangent bundle and the normal bundle are examples of *vector bundles*, which associated a vector space to each point of a vector space in a smooth way. The vector space associated to a point is called the *fibre.* An interesting example for us is a surface S embedded in 4-dimensional Euclidean space. In this case the normal bundle has 2-dimensional fibres.

Parallel transport for a vector bundle (including the tangent bundle) is the line integral of a quantity called the *connection*. We have seen that for the tangent space, parallel transport, hence the connection, is determined by two kinds properties. Let us consider next the case of the normal bundle.

Properties of the first type – that lengths of parallel vector fields and angles between parallel vector fields are preserved – continue to make sense for the normal bundle. However *torsion freeness*, i.e., that the velocity vector along a geodesic is parallel, no longer makes any sense as the velocity vector is not a normal vector.

If we require the first type of properties for parallel vector fields, then we can find connections satisfying these. But such connections are no longer unique. Nevertheless we can pick such a connection and study Differential geometry using the connection. Observe that we can still speak of the holonomy of a connection.

In general for a vector bundle we allow connections with the some properties of the first type. In the above cases these properties amounted to requiring that the holonomy is an orthogonal transformation of the fibre. Technically we say in this case that the *Gauge group* is the group of fibre-wise orthogonal transformations. We may require instead, for instance, that the Gauge group is the group of fibre-wise unitary transformations, or all fibre-wise complex linear transformations.

For any connection on a vector bundle, we can define the curvature form. From this we extract *Chern-Weil integrands*. The integrals of these are topological quantities associated to the vector bundle - its *Chern numbers*. All this is in the setting *differential geometry of fibre spaces*, developed largely by Chern.

Finally, we consider once more the question of uniqueness of connections. We saw that on the tangent bundle, there is a unique connection (called the *Levi-Civita* connection) which is torsion-free. In general we can specify properties for connections, but these do not determine a unique connection. However, there is another condition where there is a unique connection satisfying appropriate properties. This is when we have a Hermitian vector bundle on a complex space. This unique connection is the *Chern connection*.

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