DECOMPOSITION OF THE TENSOR PRODUCT OF TWO HILBERT MODULES

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This paper is dedicated to the memory of Ronald G. Douglas

ABSTRACT. Given a pair of positive real numbers α, β and a sesqui-analytic function K on a bounded domain $\Omega \subset \mathbb{C}^m$, in this paper, we investigate the properties of the sesqui-analytic function $\mathbb{K}^{(\alpha,\beta)} := K^{\alpha+\beta}\big(\partial_i\bar{\partial}_j \log K\big)_{i,j=1}^m$, taking values in $m \times m$ matrices. One of the key findings is that $\mathbb{K}^{(\alpha,\beta)}$ is non-negative definite whenever K^α and K^β are non-negative definite. In this case, a realization of the Hilbert module determined by the kernel $\mathbb{K}^{(\alpha,\beta)}$ is obtained. Let $\mathcal{M}_i$, $i = 1, 2$, be two Hilbert modules over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$. Then $\mathbb{C}[z_1, \ldots, z_{2m}]$ acts naturally on the tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$. The restriction of this action to the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ obtained using the restriction map $p \mapsto p_{|\Delta}$ leads to a natural decomposition of the tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$, which is investigated. Two of the initial pieces in this decomposition are identified.

1. Introduction

1.1. **Hilbert Module.** We will find it useful to state many of our results in the language of Hilbert modules. The notion of a Hilbert module was introduced by R. G. Douglas (cf. [11]), which we recall below. We point out that in the original definition, the module multiplication was assumed to be continuous in both the variables. However, for our purposes, it would be convenient to assume that it is continuous only in the second variable.

Definition 1.1 (Hilbert module)**.** *A Hilbert module $\mathcal{M}$ over a unital, complex algebra $\mathbb{A}$ consists of a complex Hilbert space $\mathcal{M}$ and a map $(a, h) \mapsto a \cdot h$, $a \in \mathbb{A}, h \in \mathcal{M}$, such that*

(i) $1 \cdot h = h$
(ii) $(ab) \cdot h = a \cdot (b \cdot h)$
(iii) $(a + b) \cdot h = a \cdot h + b \cdot h$
(iv) *for each a in $\mathbb{A}$, the map $\mathbf{m}_a : \mathcal{M} \to \mathcal{M}$, defined by $\mathbf{m}_a(h) = a \cdot h$, $h \in \mathcal{M}$, is a bounded linear operator on $\mathcal{M}$.*

A closed subspace $\mathcal{S}$ of $\mathcal{M}$ is said to be a submodule of $\mathcal{M}$ if $\mathbf{m}_a h \in \mathcal{S}$ for all $h \in \mathcal{S}$ and $a \in \mathbb{A}$. The quotient module $\mathcal{Q} := \mathcal{H} / \mathcal{S}$ is the Hilbert space $\mathcal{S}^\perp$, where the module multiplication is defined to be the compression of the module multiplication on $\mathcal{H}$ to the subspace $\mathcal{S}^\perp$, that is, the module action on $\mathcal{Q}$ is given by $\mathbf{m}_a(h) = P_{\mathcal{S}^\perp}(\mathbf{m}_a h)$, $h \in \mathcal{S}^\perp$. Two Hilbert modules $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathbb{A}$ are said to be isomorphic if there exists a unitary operator $U : \mathcal{M}_1 \to \mathcal{M}_2$ such that $U(a \cdot h) = a \cdot Uh$, $a \in \mathbb{A}$, $h \in \mathcal{M}_1$.

Let $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a ses-qui analytic (that is holomorphic in first m-variables and anti-holomorphic in the second set of m-variables) non-negative definite kernel on a bounded domain $\Omega \subset \mathbb{C}^m$. It uniquely determines a Hilbert space $(\mathcal{H}, K)$ consisting of holomorphic functions on Ω taking values in $\mathbb{C}^k$ possessing the following properties. For $w \in \Omega$,

(i) the vector valued function $K(\cdot, w)\zeta$, $\zeta \in \mathbb{C}^k$, belongs to the Hilbert space $\mathcal{H}$

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(ii) $\langle f, K(\cdot,w)\zeta\rangle_{\mathcal{H}} = \langle f(w),\zeta\rangle_{\mathbb{C}^k}$, $f \in (\mathcal{H},K)$.

Assume that the operator of multiplication M_{z_i} by the ith coordinate function z_i is bounded on the Hilbert space $(\mathcal{H},K)$ for $i = 1,\dots,m$. Then $(\mathcal{H},K)$ may be realized as a Hilbert module over the polynomial ring $\mathbb{C}[z_1,\dots,z_m]$ with the module action given by the point-wise multiplication:

$$\mathbf{m}_p(h) = ph,\ h \in (\mathcal{H},K),\ p \in \mathbb{C}[z_1,\dots,z_m].$$

Let K_1 and K_2 be two scalar valued non-negative definite kernels defined on $\Omega\times\Omega$. It turns out that $(\mathcal{H},K_1)\otimes(\mathcal{H},K_2)$ is the reproducing kernel Hilbert space with the reproducing kernel $K_1\otimes K_2$, where $K_1\otimes K_2 : (\Omega\times\Omega)\times(\Omega\times\Omega)\to\mathbb{C}$ is given by

$$(K_1\otimes K_2)(z,\zeta;w,\rho) = K_1(z,w)K_2(\zeta,\rho),\ \ z,\zeta,w,\rho\in\Omega.$$

Assume that the multiplication operators M_{z_i}, $i = 1,\dots,m$, are bounded on $(\mathcal{H},K_1)$ as well as on $(\mathcal{H},K_2)$. Then $(\mathcal{H},K_1)\otimes(\mathcal{H},K_2)$ may be realized as a Hilbert module over $\mathbb{C}[z_1,\dots,z_{2m}]$ with the module action defined by

$$\mathbf{m}_p(h) = ph,\ h \in (\mathcal{H},K_1)\otimes(\mathcal{H},K_2),\ p \in \mathbb{C}[z_1,\dots,z_{2m}].$$

The module $(\mathcal{H},K_1)\otimes(\mathcal{H},K_2)$ admits a natural direct sum decomposition as follows.

For a non-negative integer k, let $\mathcal{A}_k$ be the subspace of $(\mathcal{H},K_1)\otimes(\mathcal{H},K_2)$ defined by

$$\mathcal{A}_k := \big\{f \in (\mathcal{H},K_1)\otimes(\mathcal{H},K_2) : \big(\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}} f(z,\zeta)\big)_{|\Delta} = 0,\ |\boldsymbol{i}|\le k\big\}, \tag{1.1}$$

where $\boldsymbol{i}\in\mathbb{Z}_+^m$, $|\boldsymbol{i}| = i_1+\cdots+i_m$, $\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}} = \frac{\partial^{|\boldsymbol{i}|}}{\partial\zeta_1^{i_1}\cdots\partial\zeta_m^{i_m}}$, and $\big(\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}} f(z,\zeta)\big)_{|\Delta}$ is the restriction of $\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}} f(z,\zeta)$ to the diagonal set $\Delta := \{(z,z) : z\in\Omega\}$. It is easily verified that each of the subspaces $\mathcal{A}_k$ is closed and invariant under multiplication by any polynomial in $\mathbb{C}[z_1,\dots,z_{2m}]$ and therefore they are sub-modules of $(\mathcal{H},K_1)\otimes(\mathcal{H},K_2)$. Setting $\mathcal{S}_0 = \mathcal{A}_0^\perp$, $\mathcal{S}_k := \mathcal{A}_{k-1}\ominus\mathcal{A}_k$, $k = 1,2,\dots$, we obtain a direct sum decomposition of the Hilbert space

$$(\mathcal{H},K_1)\otimes(\mathcal{H},K_2) = \bigoplus_{k=0}^{\infty}\mathcal{S}_k.$$

In this decomposition, the subspaces $\mathcal{S}_k\subseteq(\mathcal{H},K_1)\otimes(\mathcal{H},K_2)$ are not necessarily sub-modules. Indeed, one may say they are semi-invariant modules following the terminology commonly used in Sz.-Nagy–Foias model theory for contractions. We study the compression of the module action to these subspaces analogous to the ones studied in operator theory. Also, such a decomposition is similar to the Clebsch-Gordan formula, which describes the decomposition of the tensor product of two irreducible representations, say ϱ_1 and ϱ_2 of a group G when restricted to the diagonal subgroup in $G\times G$:

$$\varrho_1(g)\otimes\varrho_2(g) = \bigoplus_k d_k\pi_k(g),$$

where π_k, $k\in\mathbb{Z}_+$, are irreducible representation of the group G and d_k, $k\in\mathbb{Z}_+$, are natural numbers. However, the decomposition of the tensor product of two Hilbert modules cannot be expressed as the direct sum of submodules. Noting that $\mathcal{S}_0$ is a quotient module, describing all the semi-invariant modules $\mathcal{S}_k$, $k\ge 1$, would appear to be a natural question. To describe the equivalence classes of $\mathcal{S}_0$, $\mathcal{S}_1,\dots$ etc., it would be useful to recall the notion of the push-forward of a module.

Let $\iota : \Omega\to\Omega\times\Omega$ be the map $\iota(z) = (z,z)$, $z\in\Omega$. Any Hilbert module $\mathcal{M}$ over the polynomial ring $\mathbb{C}[z_1,\dots,z_m]$ may be thought of as a module $\iota_\star\mathcal{M}$ over the ring $\mathbb{C}[z_1,\dots,z_{2m}]$ by re-defining the multiplication: $\mathbf{m}_p(h) = (p\circ\iota)h$, $h\in\mathcal{M}$ and $p\in\mathbb{C}[z_1,\dots,z_{2m}]$. The module $\iota_\star\mathcal{M}$ over $\mathbb{C}[z_1,\dots,z_{2m}]$ is defined to be the push-forward of the module $\mathcal{M}$ over $\mathbb{C}[z_1,\dots,z_m]$ under the inclusion map ι.

In [1], Aronszajn proved that the Hilbert space $(\mathcal{H},K_1K_2)$ corresponding to the point-wise product K_1K_2 of two non-negative definite kernels K_1 and K_2 is obtained by the restriction of the functions in the tensor product $(\mathcal{H},K_1)\otimes(\mathcal{H},K_2)$ to the diagonal set Δ. Building on his work, it was shown in [10] that the restriction map is isometric on the subspace $\mathcal{S}_0$ onto $(\mathcal{H},K_1K_2)$ intertwining the module actions on $\iota_\star(\mathcal{H}, K_1K_2)$ and $\mathcal{S}_0$. However, using the jet construction given below, it is possible to describe the quotient modules $\mathcal{A}_k^\perp$, $k \geq 0$. We reiterate that one of the main questions we address is that of of describing the semi-invariant modules, namely, $\mathcal{S}_1, \mathcal{S}_2, \ldots$. We have succeed in describing only $\mathcal{S}_1$ only after assuming that the pair of kernels is of the form K^α, K^β, $\alpha, \beta > 0$, where the real power of a non-negative definite kernel is defined below.

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-zero sesqui-analytic function. Let t be a real number. The function K^t is defined in the usual manner, namely $K^t(z,w) = \exp(t \log K(z,w))$, $z, w \in \Omega$, assuming that a continuous branch of the logarithm of K exists on $\Omega \times \Omega$. Clearly, K^t is also sesqui-analytic. However, if K is non-negative definite, then K^t need not be non-negative definite unless t is a natural number. A direct computation, assuming the existence of a continuous branch of logarithm of K on $\Omega \times \Omega$, shows that for $1 \leq i, j \leq m$,

$$\partial_i \bar{\partial}_j \log K(z,w) = \frac{K(z,w)\partial_i\bar{\partial}_j K(z,w) - \partial_i K(z,w) \bar{\partial}_j K(z,w)}{K(z,w)^2}, \quad z, w \in \Omega,$$

where ∂_i and $\bar{\partial}_j$ denote $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar{w}_j}$, respectively.

For a sesqui-analytic function $K : \Omega \times \Omega \to \mathbb{C}$ satisfying $K(z,z) > 0$, an alternative interpretation of $K(z,w)^t$ (resp. $\log K(z,w)$) is possible using the notion of polarization. The real analytic function $K(z,z)^t$ (resp. $\log K(z,z)$) defined on Ω extends to a unique sesqui-analytic function in some neighbourhood U of the diagonal set $\{(z,z) : z \in \Omega\}$ in $\Omega \times \Omega$. If the principal branch of logarithm of K exists on $\Omega \times \Omega$, then it is easy to verify that these two definitions of $K(z,w)^t$ (resp. $\log K(z,w)$) agree on the open set U.

In the particular case, when $K_1 = (1 - z\bar{w})^{-\alpha}$ and $K_2 = (1 - z\bar{w})^{-\beta}$, $\alpha, \beta > 0$, the description of the semi-invariant modules $\mathcal{S}_k$, $k \geq 0$, is obtained from somewhat more general results of Ferguson and Rochberg.

Theorem 1.2 (Ferguson-Rochberg,[13])**.** *If $K_1(z,w) = \frac{1}{(1-z\bar{w})^\alpha}$ and $K_2(z,w) = \frac{1}{(1-z\bar{w})^\beta}$ on $\mathbb{D} \times \mathbb{D}$ for some $\alpha, \beta > 0$, then the Hilbert modules $\mathcal{S}_n$ and $\iota_\star(\mathcal{H}, (1 - z\bar{w})^{-(\alpha+\beta+2n)})$ are isomorphic.*

In this paper, first we show that if K^α and K^β, $\alpha, \beta > 0$, are two non-negative definite kernels on Ω, then function $\mathbb{K}^{(\alpha,\beta)} : \Omega \times \Omega \to \mathcal{M}_m(\mathbb{C})$ defined by

$$\mathbb{K}^{(\alpha,\beta)}(z,w) = K^{\alpha+\beta}(z,w)\Big(\big(\partial_i \bar{\partial}_j \log K\big)(z,w) \Big)_{i,j=1}^m, \quad z, w \in \Omega,$$

is also a non-negative definite kernel. In this case, a description of the Hilbert module $\mathcal{S}_1$ is obtained. Indeed, it is shown that the Hilbert modules $\mathcal{S}_1$ and $\iota_\star\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$ are isomorphic.

1.2. **The jet construction.** For a bounded domain $\Omega \subset \mathbb{C}^m$, let K_1 and K_2 be two scalar valued non-negative kernels defined on $\Omega \times \Omega$. Assume that the multiplication operators M_{z_i}, $i = 1, \ldots, m$, are bounded on $(\mathcal{H}, K_1)$ as well as on $(\mathcal{H}, K_2)$. For a non-negative integer k, let $\mathcal{A}_k$ be the subspace defined in (1.1).

Let d be the cardinality of the set $\{\boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k\}$, which is $\binom{m+k}{m}$. Define the linear map $J_k : (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2) \to \mathrm{Hol}(\Omega \times \Omega, \mathbb{C}^{\mathrm{d}})$ by

$$(J_k f)(z,\zeta) = \sum_{|\boldsymbol{i}| \leq k} \big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z,\zeta) \otimes e_{\boldsymbol{i}}, \quad f \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2), \tag{1.2}$$

where $\{e_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k}$ is the standard orthonormal basis of $\mathbb{C}^d$. Let $R : \mathrm{ran}\ J_k \to \mathrm{Hol}(\Omega, \mathbb{C}^{\mathrm{d}})$ be the restriction map, that is, $R(\mathbf{h}) = \mathbf{h}_{|\Delta}$, $\mathbf{h} \in \mathrm{ran}\ J_k$. Clearly, $\ker RJ_k = \mathcal{A}_k$. Hence the map $RJ_k : \mathcal{A}_k^\perp \to \mathrm{Hol}(\Omega, \mathbb{C}^{\mathrm{d}})$ is one to one. Therefore we can give a natural inner product on $\mathrm{ran}\ RJ_k$, namely,

$$\langle RJ_k(f), RJ_k(g) \rangle = \langle P_{\mathcal{A}_k^\perp} f, P_{\mathcal{A}_k^\perp} g \rangle, \quad f, g \in (\mathcal{H}, K_1) \otimes (\mathcal{H}, K_2).$$

In what follows, we think of ran RJ_k as a Hilbert space equipped with this inner product. The theorem stated below is a straightforward generalization of one of the main results from [10].

Theorem 1.3. ([10, Proposition 2.3]) Let $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ be two non-negative definite kernels. Then ran RJ_k is a reproducing kernel Hilbert space and its reproducing kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$ is given by the formula

$$J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}(z, w) := \big(K_1(z, w)\partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}}K_2(z, w)\big)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^{k}, \;\; z, w \in \Omega.$$

Now for any polynomial p in z, ζ, define the operator $\mathcal{T}_p$ on ran RJ_k as

$$(\mathcal{T}p)(RJ_kf) = \sum_{|\boldsymbol{l}|\le k}\Big(\sum_{\boldsymbol{q}\le\boldsymbol{l}}\tbinom{\boldsymbol{l}}{\boldsymbol{q}}\Big(\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{q}}p(z,\zeta)\Big)_{|\Delta}\Big(\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{l}-\boldsymbol{q}}f(z,\zeta)\Big)_{|\Delta}\Big)\otimes e_{\boldsymbol{l}}, f \in (\mathcal{H}, K_1)\otimes(\mathcal{H}, K_2),$$

where $\boldsymbol{l} = (l_1, \ldots, l_m), \boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{Z}_+^m$, and $\boldsymbol{q} \le \boldsymbol{l}$ means $q_i \le l_i,\; i = 1, \ldots, m$ and $\binom{\boldsymbol{l}}{\boldsymbol{q}} = \binom{l_1}{q_1}\cdots\binom{l_m}{q_m}$. The proof of the Proposition below follows from a straightforward computation using the Leibniz rule, the details are on page 378 - 379 of [10].

Proposition 1.4. *For any polynomial p in $\mathbb{C}[z_1, \ldots, z_{2m}]$, the operator $P_{\mathcal{A}_k^\perp}M_{p|\mathcal{A}_k^\perp}$ is unitarily equivalent to the operator $\mathcal{T}_p$ on* (ran RJ_k)*.*

In section 4, we prove a generalization of the theorem of Salinas for all kernels of the form $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$. In particular, we show that if $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels (resp. generalized Bergman kernels), then so is the kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$.

In Section 5, we introduce the notion of a generalized Wallach set for an arbitrary non-negative definite kernel K defined on a bounded domain $\Omega \subset \mathbb{C}^m$. Recall that the ordinary Wallach set associated with the Bergman kernel B_Ω of a bounded symmetric domain Ω is the set $\{t > 0 :$ B_Ω^t is non-negative definite$\}$. Replacing the Bergman kernel in the definition of the Wallach set by an arbitrary non-negative definite kernel K, we define the ordinary Wallach set $\mathcal{W}(K)$. More importantly, we introduce the generalized Wallach set $G\mathcal{W}(K)$ associated to the kernel K to be the set $\{t \in \mathbb{R} :\; K^t\big(\partial_i\bar{\partial}_j \log K\big)_{i,j=1}^m$ is non-negative definite$\}$, where we have assumed that K^t is well defined for all $t \in \mathbb{R}$. In the particular case of the Euclidean unit ball $\mathbb{B}_m$ in $\mathbb{C}^m$ and the Bergman kernel, the generalized Wallach set $G\mathcal{W}(B_{\mathbb{B}_m})$, $m > 1$, is shown to be the set $\{t \in \mathbb{R} : t \ge 0\}$. If $m = 1$, then it is the set $\{t \in \mathbb{R} : t \ge -1\}$.

In Section 6, we study quasi-invariant kernels. Let $J : \mathrm{Aut}(\Omega) \times \Omega \to GL_k(\mathbb{C})$ be a function such that $J(\varphi, \cdot)$ is holomorphic for each φ in $\mathrm{Aut}(\Omega)$, where $\mathrm{Aut}(\Omega)$ is the group of all biholomorphic automorphisms of Ω. A non-negative definite kernel $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ is said to be quasi-invariant with respect to J if K satisfies the following transformation rule:

$$J(\varphi, z)K(\varphi(z), \varphi(w))J(\varphi, w)^* = K(z, w),\; z, w \in \Omega,\; \varphi \in \mathrm{Aut}(\Omega).$$

It is shown that if $K : \Omega \times \Omega \to \mathbb{C}$ is a quasi-invariant kernel with respect to $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathbb{C} \setminus \{0\}$, then the kernel $K^t\big(\partial_i\bar{\partial}_j \log K\big)_{i,j=1}^m$ is also quasi-invariant with respect to $\mathbb{J}$ whenever $t \in G\mathcal{W}(K)$, where $\mathbb{J}(\varphi, z) = J(\varphi, z)^t D\varphi(z)^{\mathrm{tr}}$, $\varphi \in \mathrm{Aut}(\Omega)$, $z \in \Omega$. In particular, taking $\Omega \subset \mathbb{C}^m$ to be a bounded symmetric domain and setting K to be the Bergman kernel B_Ω, in the language of [22], we conclude that the multiplication tuple $\boldsymbol{M}_z$ on $(\mathcal{H}, \boldsymbol{B}_\Omega^{(t)})$, where $\boldsymbol{B}_\Omega^{(t)}(z, w) := (B_\Omega^t\partial_i\bar{\partial}_j \log B_\Omega)_{i,j=1}^m$, is homogeneous with respect to the group $\mathrm{Aut}(\Omega)$ for t in $G\mathcal{W}(B_\Omega)$.

2. A new non-negative definite kernel

The scalar version of the following lemma is well-known. However, the easy modifications necessary to prove it in the case of $k \times k$ matrices are omitted.

Lemma 2.1 (Kolmogorov). *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain, and let $\mathcal{H}$ be a Hilbert space. If $\phi_1, \phi_2, \ldots, \phi_k$ are anti-holomorphic functions from Ω into $\mathcal{H}$, then $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ defined by $K(z,w) = \big(\langle \phi_j(w), \phi_i(z) \rangle_{\mathcal{H}} \big)_{i,j=1}^k$, $z, w \in \Omega$, is a sesqui-analytic non-negative definite kernel.*

For any reproducing kernel Hilbert space $(\mathcal{H}, K)$, the following proposition, which is Lemma 4.1 of [8] is a basic tool in what follows.

Proposition 2.2. *Let $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel. For every $\boldsymbol{i} \in \mathbb{Z}_+^m$, $\eta \in \mathbb{C}^k$ and $w \in \Omega$, we have*
(i) $\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\eta$ *is in* $(\mathcal{H}, K)$,
(ii) $\big\langle f, \bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\eta \big\rangle_{(\mathcal{H},K)} = \big\langle (\partial^{\boldsymbol{i}} f)(w), \eta \big\rangle_{\mathbb{C}^k}$, $f \in (\mathcal{H}, K)$.

Here and throughout this paper, for any non-negative definite kernel $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ and $\eta \in \mathbb{C}^k$, let $\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\eta$ denote the function $\big(\frac{\partial}{\partial \overline{w}_1}\big)^{i_1} \cdots \big(\frac{\partial}{\partial \overline{w}_m}\big)^{i_m} K(\cdot, w)\eta$ and $(\partial^{\boldsymbol{i}} f)(z)$ be the function $\big(\frac{\partial}{\partial z_1}\big)^{i_1} \cdots \big(\frac{\partial}{\partial z_m}\big)^{i_m} f(z)$, $\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$.

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{C}^m$ and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some α, $\beta > 0$. Then the function*
$$K^{\alpha+\beta}(z,w)\Big(\big(\partial_i \bar{\partial}_j \log K\big)(z,w) \Big)_{i,j=1}^m, \; z, w \in \Omega,$$
is a non-negative definite kernel on $\Omega \times \Omega$ taking values in $\mathcal{M}_m(\mathbb{C})$.

Proof. For $1 \le i \le m$, set $\phi_i(z) = \beta \bar{\partial}_i K^\alpha(\cdot, z) \otimes K^\beta(\cdot, z) - \alpha K^\alpha(\cdot, z) \otimes \bar{\partial}_i K^\beta(\cdot, z)$. From Proposition 2.2, it follows that each ϕ_i is a function from Ω into the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$. Then we have
$$\begin{aligned}
\langle \phi_j(w), \phi_i(z) \rangle &= \beta^2 \partial_i \bar{\partial}_j K^\alpha(z,w) K^\beta(z,w) + \alpha^2 K^\alpha(z,w) \partial_i \bar{\partial}_j K^\beta(z,w) \\
&\quad - \alpha\beta\big(\partial_i K^\alpha(z,w) \bar{\partial}_j K^\beta(z,w) + \bar{\partial}_j K^\alpha(z,w) \partial_i K^\beta(z,w)\big) \\
&= \beta^2\big(\alpha(\alpha-1)K^{\alpha+\beta-2}(z,w)\partial_i K(z,w)\bar{\partial}_j K(z,w) + \alpha K^{\alpha+\beta-1}(z,w)\partial_i\bar{\partial}_j K(z,w)\big) \\
&\quad + \alpha^2\big(\beta(\beta-1)K^{\alpha+\beta-2}(z,w)\partial_i K(z,w)\bar{\partial}_j K(z,w) + \beta K^{\alpha+\beta-1}(z,w)\partial_i\bar{\partial}_j K(z,w)\big) \\
&\qquad - 2\alpha^2\beta^2 K^{\alpha+\beta-2}(z,w)\partial_i K(z,w)\bar{\partial}_j K(z,w) \\
&= (\alpha^2\beta + \alpha\beta^2)K^{\alpha+\beta-2}(z,w)\big(K(z,w)\partial_i\bar{\partial}_j K(z,w) - \partial_i K(z,w)\bar{\partial}_j K(z,w)\big) \\
&= \alpha\beta(\alpha+\beta)K^{\alpha+\beta}(z,w)\partial_i\bar{\partial}_j \log K(z,w).
\end{aligned}$$
An application of Lemma 2.1 now completes the proof. □

The particular case, when $\alpha = 1 = \beta$ occurs repeatedly in the following. We therefore record it separately as a corollary.

Corollary 2.4. *Let Ω be a bounded domain in $\mathbb{C}^m$. If $K : \Omega \times \Omega \to \mathbb{C}$ is a non-negative definite kernel, then*
$$K^2(z,w)\big(\big(\partial_i \bar{\partial}_j \log K\big)(z,w) \big)_{i,j=1}^m$$
is also a non-negative definite kernel, defined on $\Omega \times \Omega$, taking values in $\mathcal{M}_m(\mathbb{C})$.

A more substantial corollary is the following, which is taken from [4]. Here we provide a slightly different proof. Recall that a non-negative definite kernel $K : \Omega \times \Omega \to \mathbb{C}$ is said to be *infinitely divisible* if for all $t > 0$, K^t is also non-negative definite.

Corollary 2.5. *Let Ω be a bounded domain in $\mathbb{C}^m$. Suppose that $K : \Omega \times \Omega \to \mathbb{C}$ is an infinitely divisible kernel. Then the function $\big(\big(\partial_i \bar{\partial}_j \log K\big)(z,w) \big)_{i,j=1}^m$ is a non-negative definite kernel taking values in $\mathcal{M}_m(\mathbb{C})$.*

Proof. For $t > 0$, $K^t(z,w)$ is non-negative definite by hypothesis. Then it follows, from Corollary 2.4, that $\big(K^{2t}\partial_i\bar{\partial}_j \log K^t(z,w)\big)_{i,j=1}^m$ is non-negative definite. Hence $\big(K^{2t}\partial_i\bar{\partial}_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite for all $t > 0$. Taking the limit as $t \to 0$, we conclude that $\big(\partial_i\bar{\partial}_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite. □

Remark 2.6. *It is known that even if K is a positive definite kernel, $\big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\big)_{i,j=1}^m$ need not be a non-negative definite kernel. In fact, $\big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\big)_{i,j=1}^m$ is non-negative definite if and only if K is infinitely divisible (see* [4, Theorem 3.3]*).*

Let $K : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ be the positive definite kernel given by $K(z,w) = 1 + \sum_{i=1}^\infty a_i z^i \bar{w}^i$, $z,w \in \mathbb{D}$, $a_i > 0$. For any $t > 0$, a direct computation gives

$$\begin{aligned}\big(K^t\partial\bar{\partial}\log K\big)(z,w) &= \big(1 + \sum_{i=1}^\infty a_i z^i \bar{w}^i\big)^t\, \partial\bar{\partial}\Big(\sum_{i=1}^\infty a_i z^i\bar{w}^i - \frac{(\sum_{i=1}^\infty a_i z^i \bar{w}^i)^2}{2} + \cdots\Big)\\ &= (1 + ta_1 z\bar{w} + \cdots)(a_1 + 2(2a_2 - a_1^2)z\bar{w} + \cdots)\\ &= a_1 + (4a_2 + (t-2)a_1^2)z\bar{w} + \cdots .\end{aligned}$$

Thus, if $t < 2$, one may choose $a_1, a_2 > 0$ such that $4a_2 + (t-2)a_1^2 < 0$. Hence $\big(K^t\partial\bar{\partial}\log K\big)(z,w)$ cannot be a non-negative definite kernel. Therefore, in general, for $\big(\big(K^t\partial_i\bar{\partial}_j\log K\big)(z,w)\big)_{i,j=1}^m$ to be non-negative definite, it is necessary that $t \geq 2$.

2.1. **Boundedness of the multiplication operator on $(\mathcal{H},\mathbb{K})$.** For $\alpha,\beta > 0$, let $\mathbb{K}^{(\alpha,\beta)}$ denote the kernel $K^{\alpha+\beta}(z,w)\Big(\big(\partial_i\bar{\partial}_j\log K\big)(z,w)\Big)_{i,j=1}^m$. If $\alpha = 1 = \beta$, then we write $\mathbb{K}$ instead of $\mathbb{K}^{(1,1)}$. For a holomorphic function $f : \Omega \to \mathbb{C}$, the operator M_f of multiplication by f on the linear space $\mathrm{Hol}(\Omega,\mathbb{C}^k)$ is defined by the rule $M_f h = f\,h$, $h \in \mathrm{Hol}(\Omega,\mathbb{C}^k)$, where $(f\,h)(z) = f(z)h(z)$, $z \in \Omega$. The boundedness criterion for the multiplication operator M_f restricted to the Hilbert space $(\mathcal{H},K)$ is well-known for the case of positive definite kernels. In what follows, often we have to work with a kernel which is merely non-negative definite. A precise statement is given below. The first part is from [24] and the second part follows from the observation that the boundedness of the operator $\sum_{i=1}^n M_iM_i^*$ is equivalent to the non-negative definiteness of the kernel $(c^2 - \langle z,w\rangle)K(z,w)$ for some positive constant c.

Lemma 2.7. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega\times\Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel.*

(i) *For any holomorphic function $f : \Omega \to \mathbb{C}$, the operator M_f of multiplication by f is bounded on $(\mathcal{H},K)$ if and only if there exists a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega\times\Omega$. In case M_f is bounded, $\|M_f\|$ is the infimum of all $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite.*

(ii) *The operator M_{z_i} of multiplication by the ith coordinate function z_i is bounded on $(\mathcal{H},K)$ for $i = 1,\ldots,m$, if and only if there exists a constant $c > 0$ such that $\big(c^2 - \langle z,w\rangle\big)K(z,w)$ is non-negative definite.*

As we have pointed out, the distinction between the non-negative definite kernels and the positive definite ones is very significant. Indeed, as shown in [8, Lemma 3.6], it is interesting that if the operator $\boldsymbol{M}_z := (M_{z_1},\ldots,M_{z_m})$ is bounded on $(\mathcal{H},K)$ for some non-negative definite kernel K such that $K(z,z)$, $z \in \Omega$, is invertible, then K is positive definite. A direct proof of this statement, different from the inductive proof of Curto and Salinas is in the PhD thesis of the first named author [14].

It is natural to ask if the operator M_f is bounded on $(\mathcal{H},K)$, then if it remains bounded on the Hilbert space $(\mathcal{H},\mathbb{K})$. From the Theorem stated below, in particular, it follows that the operator M_f is bounded on $(\mathcal{H},\mathbb{K})$ whenever it is bounded on $(\mathcal{H},K)$.

Theorem 2.8. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel. Let $f : \Omega \to \mathbb{C}$ be an arbitrary holomorphic function. Suppose that there exists a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega \times \Omega$. Then the function $\big(c^2 - f(z)\overline{f(w)}\big)^2\mathbb{K}(z,w)$ is non-negative definite on $\Omega \times \Omega$.*

Proof. Without loss of generality, we assume that f is non-constant and K is non-zero. The function $G(z,w) := \big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega \times \Omega$ by hypothesis. We claim that $|f(z)| < c$ for all z in Ω. If not, then by the open mapping theorem, there exists an open set $\Omega_0 \subset \Omega$ such that $|f(z)| > c$, $z \in \Omega_0$. Since $\big(c^2 - |f(z)|^2\big)K(z,z) \geq 0$, it follows that $K(z,z) = 0$ for all $z \in \Omega_0$. Now, let h be an arbitrary vector in $(\mathcal{H}, K)$. Clearly, $|h(z)| = |\langle h, K(\cdot,z)\rangle| \leq \|h\|\|K(\cdot,z)\| = \|h\|K(z,z)^{\frac{1}{2}} = 0$ for all $z \in \Omega_0$. Consequently, $h(z) = 0$ on Ω_0. Since Ω is connected and h is holomorphic, it follows that $h = 0$. This contradicts the assumption that K is non-zero verifying the validity of our claim.

From the claim, we have that the function $c^2 - f(z)\overline{f(w)}$ is non-vanishing on $\Omega \times \Omega$. Therefore, the kernel K can be written as the product

$$K(z,w) = \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)}G(z,w), \;\; z,w \in \Omega.$$

Since $|f(z)| < c$ on Ω, the function $\frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)}$ has a convergent power series expansion, namely,

$$\frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)} = \sum_{n=0}^{\infty} \frac{1}{c^{2(n+1)}} f(z)^n\overline{f(w)}^n, \;\; z,w \in \Omega.$$

Therefore it defines a non-negative definite kernel on $\Omega \times \Omega$. Note that

$$\begin{aligned}
&\big(K(z,w)^2\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m \\
&= \Big(K(z,w)^2\partial_i\bar\partial_j \log \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)}\Big)_{i,j=1}^m + \Big(K(z,w)^2\partial_i\bar\partial_j \log G(z,w)\Big)_{i,j=1}^m \\
&= \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)^2}\Big(K(z,w)^2\Big(\partial_i f(z)\overline{\partial_j f(w)}\Big)_{i,j=1}^m + G(z,w)^2\Big(\partial_i\bar\partial_j \log G(z,w)\Big)_{i,j=1}^m\Big),
\end{aligned}$$

where for the second equality, we have used that

$$\partial_i\bar\partial_j \log \frac{1}{\big(c^2 - f(z)\overline{f(w)}\big)} = \frac{\partial_i f(z)\overline{\partial_j f(w)}}{\big(c^2 - f(z)\overline{f(w)}\big)^2}, \;\; z,w \in \Omega, \; 1 \leq i,j \leq m.$$

Thus

$$\begin{aligned}
&\big(c^2 - f(z)\overline{f(w)}\big)^2\mathbb{K}(z,w) \\
&\quad = K(z,w)^2\Big(\partial_i f(z)\overline{\partial_j f(w)}\Big)_{i,j=1}^m + \Big(G(z,w)^2\partial_i\bar\partial_j \log G(z,w)\Big)_{i,j=1}^m.
\end{aligned} \tag{2.1}$$

By Lemma 2.1, the function $\big(\partial_i f(z)\overline{\partial_j f(w)}\big)_{i,j=1}^m$ is non-negative definite on $\Omega \times \Omega$. Thus the product $K(z,w)^2\big(\partial_i f(z)\overline{\partial_j f(w)}\big)_{i,j=1}^m$ is also non-negative definite on $\Omega \times \Omega$. Since G is non-negative definite on $\Omega \times \Omega$, by Corollary 2.4, the function $\big(G(z,w)^2\partial_i\bar\partial_j \log G(z,w)\big)_{i,j=1}^m$ is also non-negative definite on $\Omega \times \Omega$. The proof is now complete since the sum of two non-negative definite kernels remains non-negative definite. ∎

A sufficient condition for the boundedness of the multiplication operator on the Hilbert space $(\mathcal{H}, \mathbb{K})$ is an immediate Corollary.

Corollary 2.9. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Suppose that the multiplication operator M_f on $(\mathcal{H}, K)$ is bounded. Then the multiplication operator M_f is also bounded on $(\mathcal{H}, \mathbb{K})$.*

Proof. Since the operator M_f is bounded on $(\mathcal{H}, K)$, by Lemma 2.7, we find a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega \times \Omega$. Then, by Theorem 2.8, it follows that $\big(c^2 - f(z)\overline{f(w)}\big)^2 \mathbb{K}(z,w)$ is non-negative definite on $\Omega \times \Omega$. Also, from the proof of Theorem 2.8, we have that $\big(c^2 - f(z)\overline{f(w)}\big)^{-1}$ is non-negative definite on $\Omega \times \Omega$ (assuming that f is non-constant). Hence $\big(c - f(z)\overline{f(w)}\big)\mathbb{K}(z,w)$, being the product of two non-negative definite kernels, is non-negative definite on $\Omega \times \Omega$. An application of Lemma 2.7, a second time, completes the proof. ∎

A second Corollary provides a sufficient condition for the positive definiteness of the kernel $\mathbb{K}$.

Corollary 2.10. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel satisfying $K(w,w) > 0$, $w \in \Omega$. Suppose that the multiplication operator M_{z_i} on $(\mathcal{H}, K)$ is bounded for $i = 1, \ldots, m$. Then the kernel $\mathbb{K}$ is positive definite on $\Omega \times \Omega$.*

Proof. By Corollary 2.4, we already have that $\mathbb{K}$ is non-negative definite. Moreover, since M_{z_i} on $(\mathcal{H}, K)$ is bounded for $i = 1, \ldots, m$, it follows from Theorem 2.9 that M_{z_i} is bounded on $(\mathcal{H}, \mathbb{K})$ also. Therefore, using [8, Lemma 3.6], we see that $\mathbb{K}$ is positive definite if $\mathbb{K}(w,w)$ is invertible for all $w \in \Omega$. To verify this, set

$$\phi_i(w) = \bar{\partial}_i K(\cdot, w) \otimes K(\cdot, w) - K(\cdot, w) \otimes \bar{\partial}_i K(\cdot, w),\ 1 \le i \le m.$$

From the proof of Proposition 2.3, we see that $\mathbb{K}(w,w) = \frac{1}{2}\big(\langle \phi_j(w), \phi_i(w)\rangle\big)_{i,j=1}^m$. Therefore $\mathbb{K}(w,w)$ is invertible if the vectors $\phi_1(w), \ldots, \phi_m(w)$ are linearly independent. Note that for $w = (w_1, \ldots, w_m)$ in Ω and $j = 1, \ldots, m$, we have $(M_{z_j} - w_j)^* K(\cdot, w) = 0$. Differentiating this equation with respect to $\bar{w}_i$, we obtain

$$(M_{z_j} - w_j)^* \bar{\partial}_i K(\cdot, w) = \delta_{ij} K(\cdot, w),\ 1 \le i, j \le m.$$

Thus

$$\big((M_{z_j} - w_j)^* \otimes I\big)\big(\phi_i(w)\big) = \delta_{ij} K(\cdot, w) \otimes K(\cdot, w),\ 1 \le i, j \le m. \tag{2.2}$$

Now assume that $\sum_{i=1}^m c_i \phi_i(w) = 0$ for some scalars $c_1, \ldots, c_m$. Then, for $1 \le j \le m$, we have that $\sum_{i=1}^m \big((M_{z_j} - w_j)^* \otimes I\big)\big(\phi_i(w)\big) = 0$. Thus, using (2.2), we see that $c_j K(\cdot, w) \otimes K(\cdot, w) = 0$. Since $K(w,w) > 0$, we conclude that $c_j = 0$. Hence the vectors $\phi_1(w), \ldots, \phi_m(w)$ are linearly independent. This completes the proof. ∎

Remark 2.11. *Recall that an operator T is said to be a $2-$hyper contraction if $I - T^*T \ge 0$ and $I - 2T^*T + T^{*2}T^2 \ge 0$. If $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is a non-negative definite kernel, then it is not hard to verify that the adjoint M_z^* of the multiplication by the coordinate function z is a $2-$hyper contraction on $(\mathcal{H}, K)$ if and only if $(1 - z\bar{w})^2 K$ is non-negative definite. It follows from Theorem 2.8 that if M_z^* on $(\mathcal{H}, K)$ is a contraction, then M_z^* on $(\mathcal{H}, \mathbb{K})$ is a $2-$hyper contraction.*

3. Realization of $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^α and K^β are non-negative definite for some $\alpha, \beta > 0$. In this section, we give a description of the Hilbert space $\big(\mathcal{H},\ \mathbb{K}^{(\alpha,\beta)}\big)$. As before, we set

$$\phi_i(w) = \beta \bar{\partial}_i K^\alpha(\cdot, w) \otimes K^\beta(\cdot, w) - \alpha K^\alpha(\cdot, w) \otimes \bar{\partial}_i K^\beta(\cdot, w),\ 1 \le i \le m,\ w \in \Omega. \tag{3.1}$$

Let $\mathcal{N}$ be the subspace of $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ which is the closed linear span of the vectors

$$\big\{\, \phi_i(w) : w \in \Omega,\ 1 \le i \le m \,\big\}.$$

From the definition of $\mathcal{N}$, it is not easy to determine which vectors are in it. A useful alternative description of the space $\mathcal{N}$ is given below.

Recall that $K^\alpha \otimes K^\beta$ is the reproducing kernel for the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$, where the kernel $K^\alpha \otimes K^\beta$ on $(\Omega \times \Omega) \times (\Omega \times \Omega)$ is given by

$$K^\alpha \otimes K^\beta(z, \zeta; z', \zeta') = K^\alpha(z, z') K^\beta(\zeta, \zeta'),$$

$z = (z_1, \ldots, z_m)$, $\zeta = (\zeta_1, \ldots, \zeta_m)$, $z' = (z_{m+1}, \ldots, z_{2m})$, $\zeta' = (\zeta_{m+1}, \ldots, \zeta_{2m})$ are in Ω. We realize the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ as a space consisting of holomorphic functions on $\Omega \times \Omega$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the subspaces defined by

$$\mathcal{A}_0 = \big\{ f \in (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) : f_{|\Delta} = 0 \big\}$$

and

$$\mathcal{A}_1 = \big\{ f \in (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) : f_{|\Delta} = (\partial_{m+1} f)_{|\Delta} = \cdots = (\partial_{2m} f)_{|\Delta} = 0 \big\},$$

where Δ is the diagonal set $\{(z, z) \in \Omega \times \Omega : z \in \Omega\}$, $\partial_i f$ is the derivative of f with respect to the ith variable, and $f_{|\Delta}$, $(\partial_i f)_{|\Delta}$ denote the restrictions to the set Δ of the functions f, $\partial_i f$, respectively. It is easy to see that both $\mathcal{A}_0$ and $\mathcal{A}_1$ are closed subspaces of the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ and $\mathcal{A}_1$ is a closed subspace of $\mathcal{A}_0$.

Now observe that, for $1 \le i \le m$, we have

$$\begin{aligned} \bar\partial_i (K^\alpha \otimes K^\beta)(\cdot, (z', \zeta')) &= \bar\partial_i K^\alpha(\cdot, z') \otimes K^\beta(\cdot, \zeta'),\ z', \zeta' \in \Omega \\ \bar\partial_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (z', \zeta')) &= K^\alpha(\cdot, z') \otimes \bar\partial_i K^\beta(\cdot, \zeta'),\ z', \zeta' \in \Omega. \end{aligned} \tag{3.2}$$

Hence, taking $z' = \zeta' = w$, we see that

$$\phi_i(w) = \beta \bar\partial_i (K^\alpha \otimes K^\beta)(\cdot, (w, w)) - \alpha \bar\partial_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (w, w)). \tag{3.3}$$

We now state a useful lemma on the Taylor coefficients of an analytic functions. The straightforward proof follows from the chain rule [25, page 8], which is omitted.

Lemma 3.1. *Suppose that $f : \Omega \times \Omega \to \mathbb{C}$ is a holomorphic function satisfying $f_{|\Delta} = 0$. Then*

$$(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0, \quad 1 \le i \le m.$$

An alternative description of the subspace $\mathcal{N}$ of $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ is provided below.

Proposition 3.2. $\mathcal{N} = \mathcal{A}_0 \ominus \mathcal{A}_1$.

Proof. For all $z \in \Omega$, we see that

$$\phi_i(w)(z, z) = \alpha\beta K^{\alpha+\beta-1}(z, w) \bar\partial_i K(z, w) - \alpha\beta K^{\alpha+\beta-1}(z, w) \bar\partial_i K(z, w) = 0.$$

Hence each $\phi_i(w)$, $w \in \Omega, 1 \le i \le m$, belongs to $\mathcal{A}_0$ and consequently, $\mathcal{N} \subset \mathcal{A}_0$. Therefore, to complete the proof of the proposition, it is enough to show that $\mathcal{A}_0 \ominus \mathcal{N} = \mathcal{A}_1$.

To verify this, note that $f \in \mathcal{N}^\perp$ if and only if $\langle f, \phi_i(w) \rangle = 0$, $1 \le i \le m$, $w \in \Omega$. Now, in view of (3.3) and Proposition 2.2, we have that

$$\begin{aligned} \langle f, \phi_i(w) \rangle =& \Big\langle f, \beta \bar\partial_i (K^\alpha \otimes K^\beta)(\cdot, (w, w)) - \alpha \bar\partial_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (w, w)) \Big\rangle \\ =& \beta (\partial_i f)(w, w) - \alpha (\partial_{m+i} f)(w, w),\ 1 \le i \le m,\ w \in \Omega. \end{aligned} \tag{3.4}$$

Thus $f \in \mathcal{N}^\perp$ if and only if the function $\beta\, (\partial_i f)_{|\Delta} - \alpha\, (\partial_{m+i} f)_{|\Delta} = 0$, $1 \le i \le m$. Combining this with Lemma 3.1, we see that any $f \in \mathcal{A}_0 \ominus \mathcal{N}$, satisfies

$$\begin{aligned} \beta (\partial_i f)_{|\Delta} - \alpha (\partial_{m+i} f)_{|\Delta} &= 0, \\ (\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} &= 0, \end{aligned}$$

for $1 \le i \le m$. Therefore, we have $(\partial_i f)_{|\Delta} = (\partial_{m+i} f)_{|\Delta} = 0$, $1 \le i \le m$. Hence f belongs to $\mathcal{A}_1$.

Conversely, let $f \in \mathcal{A}_1$. In particular, $f \in \mathcal{A}_0$. Hence invoking Lemma 3.1 once again, we see that

$$(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0,\ 1 \le i \le m.$$

Since f is in $\mathcal{A}_1$, $(\partial_{m+i}f)_{|\Delta} = 0$, $1 \le i \le m$, by definition. Therefore, $(\partial_i f)_{|\Delta} = (\partial_{m+i}f)_{|\Delta} = 0$, $1 \le i \le m$, which implies

$$\beta(\partial_i f)_{|\Delta} - \alpha(\partial_{m+i}f)_{|\Delta} = 0,\ 1 \le i \le m.$$

Hence $f \in \mathcal{A}_0 \ominus \mathcal{N}$, completing the proof. □

We now give a description of the Hilbert space $\big(\mathcal{H},\, \mathbb{K}^{(\alpha,\beta)}\big)$. Define a linear map $\mathcal{R}_1 : (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta) \to \mathrm{Hol}(\Omega, \mathbb{C}^m)$ by setting

$$\mathcal{R}_1(f) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} (\beta\partial_1 f - \alpha\partial_{m+1} f)_{|\Delta} \\ \vdots \\ (\beta\partial_m f - \alpha\partial_{2m} f)_{|\Delta} \end{pmatrix} \tag{3.5}$$

for $f \in (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ and note that

$$\mathcal{R}_1(f)(w) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} \langle f, \phi_1(w)\rangle \\ \vdots \\ \langle f, \phi_m(w)\rangle \end{pmatrix},\ w \in \Omega. \tag{3.6}$$

From Equation (3.6), it is easy to see that $\ker \mathcal{R}_1 = \mathcal{N}^\perp$. We have $\mathcal{N} = \mathcal{A}_0 \ominus \mathcal{A}_1$, see Proposition 3.2. Therefore, $\ker \mathcal{R}_1^\perp = \mathcal{A}_0 \ominus \mathcal{A}_1$ and the map $\mathcal{R}_{1|\mathcal{A}_0\ominus\mathcal{A}_1} \to \mathrm{ran}\ \mathcal{R}_1$ is bijective. Require this map to be a unitary by defining an appropriate inner product on ran $\mathcal{R}_1$, that is, Set

$$\langle \mathcal{R}_1(f), \mathcal{R}_1(g)\rangle := \langle P_{\mathcal{A}_0\ominus\mathcal{A}_1} f, P_{\mathcal{A}_0\ominus\mathcal{A}_1} g\rangle,\ f, g \in (\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta), \tag{3.7}$$

where $P_{\mathcal{A}_0\ominus\mathcal{A}_1}$ is the orthogonal projection of $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ onto the subspace $\mathcal{A}_0 \ominus \mathcal{A}_1$. This choice of the inner product on the range of $\mathcal{R}_1$ makes the map $\mathcal{R}_1$ unitary.

Theorem 3.3. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^α and K^β are non-negative definite for some $\alpha, \beta > 0$. Let $\mathcal{R}_1$ be the map defined by* (3.5)*. Then the Hilbert space determined by the non-negative definite kernel $\mathbb{K}^{(\alpha,\beta)}$ coincides with the space* ran $\mathcal{R}_1$ *and the inner product given by* (3.7) *on* ran $\mathcal{R}_1$ *agrees with the one induced by the kernel $\mathbb{K}^{(\alpha,\beta)}$.*

Proof. Let $\{e_1, \dots, e_m\}$ be the standard orthonormal basis of $\mathbb{C}^m$. For $1 \le i, j \le m$, from the proof of Proposition 2.3, we have

$$\langle \phi_j(w), \phi_i(z)\rangle = \alpha\beta(\alpha+\beta) K^{\alpha+\beta}(z,w)\partial_i\bar{\partial}_j \log K(z,w) \tag{3.8}$$

$$= \alpha\beta(\alpha+\beta)\left\langle \mathbb{K}^{(\alpha,\beta)}(z,w)e_j, e_i\right\rangle_{\mathbb{C}^m},\ z, w \in \Omega. \tag{3.9}$$

Therefore, from (3.6), it follows that for all $w \in \Omega$ and $1 \le j \le m$,

$$\mathcal{R}_1(\phi_j(w)) = \sqrt{\alpha\beta(\alpha+\beta)}\,\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j.$$

Hence, for all $w \in \Omega$ and $\eta \in \mathbb{C}^m$, $\mathbb{K}^{(\alpha,\beta)}(\cdot, w)\eta$ belongs to ran $\mathcal{R}_1$. Let $\mathcal{R}_1(f)$ be an arbitrary element in ran $\mathcal{R}_1$ where $f \in \mathcal{A}_0 \ominus \mathcal{A}_1$. Then

$$\begin{aligned} \left\langle \mathcal{R}_1(f), \mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j\right\rangle &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\langle \mathcal{R}_1(f), \mathcal{R}_1(\phi_j(w))\rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\langle f, \phi_j(w)\rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}(\beta\partial_j f(w,w) - \alpha\partial_{m+j} f(w,w)) \\ &= \langle \mathcal{R}_1(f)(w), e_j\rangle_{\mathbb{C}^m}, \end{aligned}$$

where the second equality follows since both f and $\phi_j(w)$ belong to $\mathcal{A}_0 \ominus \mathcal{A}_1$. This completes the proof. □

We obtain the density of polynomials in $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$ as a consequence of this theorem. Let $\boldsymbol{z} = (z_1, \ldots, z_m)$ and let $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_m]$ denote the ring of polynomials in m-variables. The following proposition gives a sufficient condition for density of $\mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$ in the Hilbert space $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$.

Proposition 3.4. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that both the Hilbert spaces $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$ contain the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ as a dense subset. Then the Hilbert space $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$ contains the ring $\mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$ as a dense subset.*

Proof. Since $C[\boldsymbol{z}]$ is dense in both the Hilbert spaces $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$, it follows that $C[\boldsymbol{z}] \otimes C[\boldsymbol{z}]$, which is $\mathbb{C}[z_1, \ldots, z_{2m}]$, is contained in the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ and is dense in it. Since $\mathcal{R}_1$ maps $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ onto $\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$, to complete the proof, it suffices to show that $\mathcal{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) = \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$. It is easy to see that $\mathcal{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) \subseteq \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$. Conversely, if $\sum_{i=1}^m p_i(z_1, \ldots, z_m) \otimes e_i$ is an arbitrary element of $\mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$, then it is easily verified that the function $p(z_1, \ldots, z_{2m}) := \sqrt{\frac{\alpha\beta}{\alpha+\beta}} \sum_{i=1}^m (z_i - z_{m+i}) p_i(z_1, \ldots, z_m)$ belongs to $\mathbb{C}[z_1, \ldots, z_{2m}]$ and $\mathcal{R}_1(p) = \sum_{i=1}^m p_i(z_1, \ldots, z_m) \otimes e_i$. Therefore $\mathcal{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) = \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$, completing the proof. ☐

3.1. **Description of the Hilbert module $\mathcal{S}_1$.** In this subsection, we give a description of the Hilbert module $\mathcal{S}_1$ in the particular case when $K_1 = K^\alpha$ and $K_2 = K^\beta$ for some sesqui-analytic function K defined on $\Omega \times \Omega$ and a pair of positive real numbers α, β.

Theorem 3.5. *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$. Suppose that the multiplication operators $M_{z_i}, i = 1, 2, \ldots, m$, are bounded on both $(\mathcal{H}, K^\alpha)$ and $(\mathcal{H}, K^\beta)$. Then the Hilbert module $\mathcal{S}_1$ is isomorphic to the push-forward module $\iota_\star\big(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)}\big)$ via the module map $\mathcal{R}_{1|\mathcal{S}_1}$.*

Proof. From Theorem 3.3, it follows that the map $\mathcal{R}_1$ defined in (3.5) is a unitary map from $\mathcal{S}_1$ onto $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. Now we will show that $\mathcal{R}_1 P_{\mathcal{S}_1}(ph) = (p \circ \iota)\mathcal{R}_1 h,\ h \in \mathcal{S}_1, p \in \mathbb{C}[z_1, \ldots, z_{2m}]$. Let h be an arbitrary element of $\mathcal{S}_1$. Since $\ker \mathcal{R}_1 = \mathcal{S}_1^\perp$ (see the discussion before Theorem 3.3), it follows that $\mathcal{R}_1 P_{\mathcal{S}_1}(ph) = \mathcal{R}_1(ph),\ p \in \mathbb{C}[z_1, \ldots, z_{2m}]$. Hence

$$
\begin{aligned}
\mathcal{R}_1 P_{\mathcal{S}_1}(ph) &= \mathcal{R}_1(ph)\\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \sum_{j=1}^m (\beta\partial_j(ph) - \alpha\partial_{m+j}(ph))_{|\Delta} \otimes e_j\\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \sum_{j=1}^m p_{|\Delta}(\beta\partial_j h - \alpha\partial_{m+j} h)_{|\Delta} \otimes e_j + \sum_{j=1}^m h_{|\Delta}(\beta\partial_j p - \alpha\partial_{m+j} p)_{|\Delta} \otimes e_j\\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \sum_{j=1}^m p_{|\Delta}(\beta\partial_j h - \alpha\partial_{m+j} h)_{|\Delta} \otimes e_j \ \ (\text{since } h \in \mathcal{S}_1)\\
&= (p \circ \iota)\mathcal{R}_1 h,
\end{aligned}
$$

completing the proof. ☐

Notation 3.6. *For $1 \le i \le m$, let $M_i^{(1)}$ and $M_i^{(2)}$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathcal{H}, K_1)$ and $(\mathcal{H}, K_2)$, respectively. If $m = 1$, we let $M^{(1)}$ and $M^{(2)}$ denote the operators $M_1^{(1)}$ and $M_1^{(2)}$, respectively.*

In case $K_1 = K^\alpha$ and $K_2 = K^\beta$, let $M_i^{(\alpha)}$, $M_i^{(\beta)}$ and $M_i^{(\alpha+\beta)}$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathcal{H}, K^\alpha)$, $(\mathcal{H}, K^\beta)$ and $(\mathcal{H}, K^{\alpha+\beta})$, respectively. If $m = 1$, we write $M^{(\alpha)}$, $M^{(\beta)}$ and $M^{(\alpha+\beta)}$ instead of $M_1^{(\alpha)}$, $M_1^{(\beta)}$ and $M_1^{(\alpha+\beta)}$, respectively.

Finally, let $\mathbb{M}_i^{(\alpha,\beta)}$ denote the operator of multiplication by the coordinate function z_i on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. Also let $\mathbb{M}^{(\alpha,\beta)}$ denote the operator $\mathbb{M}_1^{(\alpha,\beta)}$ whenever $m = 1$.

Remark 3.7. *It is verified that* $(M_i^{(\alpha)}\otimes I)^*(\phi_j(w))=\bar{w}_i\phi_j(w)+\beta\delta_{ij}K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)$ *and* $(I\otimes M_i^{(\beta)})^*(\phi_j(w))=\bar{w}_i\phi_j(w)-\alpha\delta_{ij}K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)$, $1\le i,j\le m, w\in\Omega$. *Therefore,*

$$P_{\mathcal{S}_1}(M_i^{(\alpha)}\otimes I)_{|\mathcal{S}_1}=P_{\mathcal{S}_1}(I\otimes M_i^{(\beta)})_{|\mathcal{S}_1},\ i=1,2,\ldots,m.$$

Corollary 3.8. *The m-tuple of operators* $\big(P_{\mathcal{S}_1}(M_1^{(\alpha)}\otimes I)_{|\mathcal{S}_1},\ldots,P_{\mathcal{S}_1}(M_m^{(\alpha)}\otimes I)_{|\mathcal{S}_1}\big)$ *is unitarily equivalent to the m-tuple of operators* $(\mathbb{M}_1^{(\alpha,\beta)},\ldots,\mathbb{M}_m^{(\alpha,\beta)})$ *on* $\big(\mathcal{H},\mathbb{K}^{(\alpha,\beta)}\big)$. *In particular, if either the m-tuple of operators* $(M_1^{(\alpha)},\ldots,M_m^{(\alpha)})$ *on* $(\mathcal{H},K^\alpha)$ *or the m-tuple of operators* $(M_{(1)}^{(\beta)},\ldots,M_m^{(\beta)})$ *on* $(\mathcal{H},K^\beta)$ *is bounded, then the m-tuple* $(\mathbb{M}_1^{(\alpha,\beta)},\ldots,\mathbb{M}_m^{(\alpha,\beta)})$ *is also bounded on* $\big(\mathcal{H},\mathbb{K}^{(\alpha,\beta)}\big)$.

Proof. The proof of the first statement follows from Theorem 3.5 and the proof of the second statement follows from the first together with Remark 3.7. ☐

3.2. **Description of the quotient module $\mathcal{A}_1^\perp$.** In this subsection, we give a description of the quotient module $\mathcal{A}_1^\perp$. Let $(\mathcal{H},K^{\alpha+\beta})\widehat{\oplus}(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ be the Hilbert module, which is the Hilbert space $(\mathcal{H},K^{\alpha+\beta})\oplus(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ equipped with the multiplication over the polynomial ring $\mathbb{C}[z_1,\ldots,z_{2m}]$ induced by the $2m$-tuple of operators $(T_1,\ldots,T_m,T_{m+1},\ldots,T_{2m})$ described below. First, for any polynomial $p\in\mathbb{C}[z_1,\ldots,z_{2m}]$, let $p^*(z):=(p\circ\iota)(z)=p(z,z)$, $z\in\Omega$ and let $S_p:(\mathcal{H},K^{\alpha+\beta})\to(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ be the operator given by

$$S_p(f_0)=\frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\sum_{j=1}^m(\beta(\partial_jp)^*-\alpha(\partial_{m+j}p)^*)f_0\otimes e_j, f_0\in(\mathcal{H},K^{\alpha+\beta}).$$

On the Hilbert space $(\mathcal{H},K^{\alpha+\beta})\oplus(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$, let $T_i=\begin{pmatrix}M_{z_i}&0\\S_{z_i}&M_{z_i}\end{pmatrix}$, and $T_{m+i}=\begin{pmatrix}M_{z_i}&0\\S_{z_{m+i}}&M_{z_i}\end{pmatrix}$, $1\le i\le m$. Now, a straightforward verification shows that the module multiplication induced by these $2m$-tuple of operators is given by the formula:

$$\mathbf{m}_p(f_0\oplus f_1)=\begin{pmatrix}M_{p^*}f_0&0\\S_pf_0&M_{p^*}f_1\end{pmatrix},\ f_0\oplus f_1\in(\mathcal{H},K^{\alpha+\beta})\oplus(\mathcal{H},\mathbb{K}^{(\alpha,\beta)}).\tag{3.10}$$

Clearly, this module multiplication is distinct from the one induced by the $M_p\oplus M_p$, $p\in\mathbb{C}[z_1,\ldots,z_m]$ on the direct sum $(\mathcal{H},K^{\alpha+\beta})\oplus(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$.

Theorem 3.9. *Let* $K:\Omega\times\Omega\to\mathbb{C}$ *be a sesqui-analytic function such that the functions* K^α *and* K^β, *defined on* $\Omega\times\Omega$, *are non-negative definite for some* $\alpha,\beta>0$. *Suppose that the multiplication operators* $M_{z_i}, i=1,2,\ldots,m$, *are bounded on both* $(\mathcal{H},K^\alpha)$ *and* $(\mathcal{H},K^\beta)$. *Then the quotient module* $\mathcal{A}_1^\perp$ *and the Hilbert module* $(\mathcal{H},K^{\alpha+\beta})\widehat{\oplus}(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ *are isomorphic.*

Proof. The proof is accomplished by showing that the compression operator $P_{\mathcal{A}_1^\perp}M_{p|\mathcal{A}_1^\perp}$ is unitarily equivalent to the operator $\begin{pmatrix}M_{p^*}&0\\S_p&M_{p^*}\end{pmatrix}$ on $(\mathcal{H},K^{\alpha+\beta})\bigoplus(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ for an arbitrary polynomial p in $\mathbb{C}[z_1,\ldots,z_{2m}]$.

We recall that the map $\mathcal{R}_0:(\mathcal{H},K^\alpha)\otimes(\mathcal{H},K^\beta)\to(\mathcal{H},K^{\alpha+\beta})$ given by $\mathcal{R}_0(f)=f_{|\Delta}$, f in $(\mathcal{H},K^\alpha)\otimes(\mathcal{H},K^\beta)$ defines a unitary map from $\mathcal{S}_0$ onto $(\mathcal{H},K^{\alpha+\beta})$, and it intertwines the operators $P_{\mathcal{S}_0}M_{p|\mathcal{S}_0}$ on $\mathcal{S}_0$ and M_{p^*} on $(\mathcal{H},K^{\alpha+\beta})$, that is, $M_{p^*}\mathcal{R}_{0|\mathcal{S}_0}=\mathcal{R}_{0|\mathcal{S}_0}P_{\mathcal{S}_0}M_{p|\mathcal{S}_0}$. Combining this with Theorem 3.3, we conclude that the map $\mathcal{R}=\begin{pmatrix}\mathcal{R}_{0|\mathcal{S}_0}&0\\0&\mathcal{R}_{1|\mathcal{S}_1}\end{pmatrix}$ is unitary from $\mathcal{S}_0\bigoplus\mathcal{S}_1$ (which is $\mathcal{A}_1^\perp$) to $(\mathcal{H},K^{\alpha+\beta})\bigoplus(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$. Since $\mathcal{S}_0$ is invariant under M_p^*, it follows that $P_{\mathcal{S}_1}M^*_{p\,|\mathcal{S}_0}=0$. Hence

$$\mathcal{R}P_{\mathcal{A}_1^\perp}M^*_{p\,|\mathcal{A}_1^\perp}\mathcal{R}^*=\begin{pmatrix}\mathcal{R}_0P_{\mathcal{S}_0}M^*_{p\,|\mathcal{S}_0}\mathcal{R}_0^*&\mathcal{R}_0P_{\mathcal{S}_0}M^*_{p\,|\mathcal{S}_1}\mathcal{R}_1^*\\0&\mathcal{R}_1P_{\mathcal{S}_1}M^*_{p\,|\mathcal{S}_1}\mathcal{R}_1^*\end{pmatrix}$$

on $\mathcal{S}_0 \bigoplus \mathcal{S}_1$. We have $\mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p\,|\mathcal{S}_0} \mathcal{R}_0^* = (M_{p^*})^*$, already, on $(\mathcal{H}, K^{\alpha+\beta})$. From Theorem 3.5, we see that $\mathcal{R}_1 P_{\mathcal{S}_1} M^*_{p\,|\mathcal{S}_1} \mathcal{R}_1^* = (M_{p^*})^*$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. To prove this, note that $\mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p\,|\mathcal{S}_1} \mathcal{R}_1^* = S_p^*$. Recall that $\mathcal{R}_1^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j) = \phi_j(w)$. Consequently, an easy computation gives

$$\mathcal{R}_0 P_{\mathcal{S}_0} M^*_{p\,|\mathcal{S}_1} \mathcal{R}_1^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)}) K^{\alpha+\beta}(\cdot, w).$$

Set $S_p^\sharp = \mathcal{R}_1 P_{\mathcal{S}_1} M_{p|\mathcal{S}_0} \mathcal{R}_0^*$. Then for $1 \le j \le m$, and $w \in \Omega$, we get

$$(S_p^\sharp)^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)}) K^{\alpha+\beta}(\cdot, w).$$

For f in $(\mathcal{H}, K^{\alpha+\beta})$, we have

$$\begin{aligned}
\langle S_p^\sharp f(z), e_j \rangle &= \langle S_p^\sharp f, \mathbb{K}^{(\alpha,\beta)}(\cdot, z)e_j \rangle \\
&= \langle f, (S_p^\sharp)^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, z)e_j) \rangle \\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\beta(\partial_j p)(z,z) - \alpha(\partial_{m+j} p)(z,z)) \langle f, K^{\alpha+\beta}(\cdot, z) \rangle \\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \big(\beta(\partial_j p)(z,z) - \alpha(\partial_{m+j} p)(z,z)\big) f(z).
\end{aligned}$$

Hence $S_p^\sharp = S_p$, completing the proof of the theorem. □

Corollary 3.10. *Let $\Omega \subset \mathbb{C}$ be a bounded domain. The operator $P_{\mathcal{A}_1^\perp}(M^{(\alpha)} \otimes I)_{|\mathcal{A}_1^\perp}$ is unitarily equivalent to the operator $\begin{pmatrix} M^{(\alpha+\beta)} & 0 \\ \delta\, \mathsf{inc} & \mathbb{M}^{(\alpha,\beta)} \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha+\beta}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$, where $\delta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and* inc *is the inclusion operator from $(\mathcal{H}, K^{\alpha+\beta})$ into $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$.*

4. Generalized Bergman Kernels

We now discuss an important class of operators introduced by Cowen and Douglas in the very influential paper [6]. The case of 2 variables was discussed in [7], while a detailed study in the general case appeared later in [8]. The definition below is taken from [8]. Let $\boldsymbol{T} := (T_1, ..., T_m)$ be a m-tuple of commuting bounded linear operators on a separable Hilbert space $\mathcal{H}$. Let $D_{\boldsymbol{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ be the operator defined by $D_{\boldsymbol{T}}(x) = (T_1 x, ..., T_m x),\ x \in \mathcal{H}$.

Definition 4.1 (Cowen-Douglas class operator)**.** *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. The operator $\boldsymbol{T}$ is said to be in the Cowen-Douglas class $B_n(\Omega)$ if $\boldsymbol{T}$ satisfies the following requirements:*

(i) $\dim \ker D_{\boldsymbol{T}-w} = n,\ w \in \Omega$
(ii) ran $D_{\boldsymbol{T}-w}$ is closed for all $w \in \Omega$
(iii) $\bigvee \big\{ \ker D_{\boldsymbol{T}-w} : w \in \Omega \big\} = \mathcal{H}$.

If $\boldsymbol{T} \in B_n(\Omega)$, then for each $w \in \Omega$, there exist functions $\gamma_1, \ldots, \gamma_n$ holomorphic in a neighbourhood $\Omega_0 \subseteq \Omega$ containing w such that $\ker D_{\boldsymbol{T}-w'} = \bigvee\{\gamma_1(w'), \ldots, \gamma_n(w')\}$ for all $w' \in \Omega_0$ (cf. [7]). Consequently, every $\boldsymbol{T} \in B_n(\Omega)$ corresponds to a rank n holomorphic hermitian vector bundle $E_{\boldsymbol{T}}$ defined by

$$E_{\boldsymbol{T}} = \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker D_{\boldsymbol{T}-w}\}$$

and $\pi(w, x) = w$, $(w, x) \in E_{\boldsymbol{T}}$.

For a bounded domain Ω in $\mathbb{C}^m$, let $\Omega^* = \{z : \bar{z} \in \Omega\}$. It is known that if T is an operator in $B_n(\Omega^*)$, then for each $w \in \Omega$, T is unitarily equivalent to the adjoint of the multiplication tuple $(M_{z_1}, \ldots, M_{z_m})$ on some reproducing kernel Hilbert space $(\mathcal{H}, K) \subseteq \mathrm{Hol}(\Omega_0, \mathbb{C}^n)$ for some open subset $\Omega_0 \subseteq \Omega$ containing w. Here the kernel K can be described explicitly as follows. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$

be a holomorphic frame of the vector bundle $E_{\boldsymbol{T}}$ on a neighbourhood $\Omega_0^* \subseteq \Omega^*$ containing $\bar{w}$. Define $K_\Gamma : \Omega_0 \times \Omega_0 \to \mathcal{M}_n(\mathbb{C})$ by $K_\Gamma(z,w) = \big(\langle \gamma_j(\bar{w}), \gamma_i(\bar{z})\rangle\big)_{i,j=1}^n$, $z,w \in \Omega_0$. Setting $K = K_\Gamma$, one may verify that the operator $\boldsymbol{T}$ is unitarily equivalent to the adjoint of the m-tuple of multiplication operators $(M_{z_1},\ldots,M_{z_m})$ on the Hilbert space $(\mathcal{H},K)$.

If $T \in B_1(\Omega^*)$, the curvature matrix $\mathcal{K}_T(\bar{w})$ at a fixed but arbitrary point $\bar{w} \in \Omega^*$ is defined by

$$\mathcal{K}_T(\bar{w}) = \big(\partial_i \bar{\partial}_j \log \|\gamma(\bar{w})\|^2\big)_{i,j=1}^m,$$

where γ is a holomorphic frame of E_T defined on some open subset $\Omega_0^* \subseteq \Omega^*$ containing $\bar{w}$. If T is realized as the adjoint of the multiplication tuple $(M_{z_1},\ldots,M_{z_m})$ on some reproducing kernel Hilbert space $(\mathcal{H},K) \subseteq \mathrm{Hol}(\Omega_0)$, where $w \in \Omega_0$, the curvature $\mathcal{K}_T(\bar{w})$ is then equal to

$$\big(\partial_i \bar{\partial}_j \log K(w,w)\big)_{i,j=1}^m.$$

The study of operators in the Cowen-Douglass class using the properties of the kernel functions was initiated by Curto and Salinas in [8]. The following definition is taken from [26].

Definition 4.2 (Sharp kernel and generalized Bergman kernel)**.** *A positive definite kernel $K : \Omega\times\Omega \to \mathcal{M}_k(\mathbb{C})$ is said to be sharp if*

(i) *the multiplication operator M_{z_i} is bounded on $(\mathcal{H},K)$ for $i = 1,\ldots,m$,*
(ii) $\ker D_{(\boldsymbol{M}_z - w)^*} = \mathrm{ran}\ K(\cdot,w)$, $w \in \Omega$,

where $\boldsymbol{M}_z$ denotes the m-tuple $(M_{z_1}, M_{z_2},\ldots,M_{z_m})$ on $(\mathcal{H},K)$. Moreover, if $\mathrm{ran}\ D_{(\boldsymbol{M}_z - w)^*}$ *is closed for all $w \in \Omega$, then K is said to be a generalized Bergman kernel.*

We start with the following lemma (cf. [9, page 285]) which provides a sufficient condition for the sharpness of a non-negative definite kernel K.

Lemma 4.3. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega\times\Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel. Assume that the multiplication operator M_{z_i} on $(\mathcal{H},K)$ is bounded for $1 \le i \le m$. If the vector valued polynomial ring $\mathbb{C}[z_1,\ldots,z_m]\otimes\mathbb{C}^k$ is contained in $(\mathcal{H},K)$ as a dense subset, then K is a sharp kernel.*

Corollary 4.4. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega\times\Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega\times\Omega$ for some $\alpha,\beta > 0$. Suppose that either the m-tuple of operators $(M_1^{(\alpha)},\ldots,M_m^{(\alpha)})$ on $(\mathcal{H},K^\alpha)$ or the m-tuple of operators $(M_1^{(\beta)},\ldots,M_m^{(\beta)})$ on $(\mathcal{H},K^\beta)$ is bounded. If both the Hilbert spaces $(\mathcal{H},K^\alpha)$ and $(\mathcal{H},K^\beta)$ contain the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$ as a dense subset, then the kernel $\mathbb{K}^{(\alpha,\beta)}$ is sharp.*

Proof. By Corollary 3.8, we have that the m-tuple of operators $(\mathbb{M}_1^{(\alpha,\beta)},\ldots,\mathbb{M}_m^{(\alpha,\beta)})$ is bounded on $\big(\mathcal{H},\mathbb{K}^{(\alpha,\beta)}\big)$. If both the Hilbert spaces $(\mathcal{H},K^\alpha)$ and $(\mathcal{H},K^\beta)$ contain the polynomial ring $\mathbb{C}[z_1,\ldots,z_m]$ as a dense subset, then by Proposition 3.4, we see that the ring $\mathbb{C}[z_1,\ldots,z_m]\otimes\mathbb{C}^m$ is contained in $(\mathcal{H},\mathbb{K}^{(\alpha,\beta)})$ and is dense in it. An application of Lemma 4.3 now completes the proof. ☐

Some of the results in this paper generalize, among other things, one of the main results of [26], which is reproduced below.

Theorem 4.5 (Salinas, [26, Theorem 2.6])**.** *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega\times\Omega \to \mathbb{C}$ are two sharp kernels (resp. generalized Bergman kernels), then $K_1 \otimes K_2$ and K_1K_2 are also sharp kernels (resp. generalized Bergman kernels).*

For two scalar valued non-negative definite kernels K_1 and K_2, defined on $\Omega\times\Omega$, the jet construction (Theorem 1.3) gives rise to a family of non-negative kernels $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}$, $k \ge 0$, where

$$J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(z,w) := \big(K_1(z,w)\partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}} K_2(z,w)\big)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^k,\quad z,w \in \Omega.$$

In the particular case when $k = 0$, it coincides with the point-wise product K_1K_2. In this section, we generalize Theorem 4.5 for all kernels of the form $J_k(K_1, K_2)_{|\text{res}\,\Delta}$. First, we discuss two important corollaries of the jet construction which will be used later in this paper.

For $1 \le i \le m$, let J_kM_i denote the operator of multiplication by the ith coordinate function z_i on the Hilbert space $\big(\mathcal{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta}\big)$. In case $m = 1$, we write J_kM instead of J_kM_1.

Taking $p(z, \zeta)$ to be the ith coordinate function z_i in Proposition 1.4, we obtain the following corollary.

Corollary 4.6. *Let $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ be two non-negative definite kernels. Then the m-tuple of operators $\big(P_{\mathcal{A}_k^\perp}(M_1^{(1)} \otimes I)_{|\mathcal{A}_k^\perp}, \ldots, P_{\mathcal{A}_k^\perp}(M_m^{(1)} \otimes I)_{|\mathcal{A}_k^\perp}\big)$ is unitarily equivalent to the m-tuple $(J_kM_1, \ldots, J_kM_m)$ on the Hilbert space $\big(\mathcal{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta}\big)$.*

Combining this with Corollary 3.10 we obtain the following result.

Corollary 4.7. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. The following operators are unitarily equivalent:*

(i) *the operator $P_{\mathcal{A}_1^\perp}(M^{(\alpha)} \otimes I)_{|\mathcal{A}_1^\perp}$*
(ii) *the multiplication operator J_1M on $\big(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta}\big)$*
(iii) *the operator $\begin{pmatrix} M^{(\alpha+\beta)} & 0 \\ \delta\, \mathsf{inc} & \mathbb{M}^{(\alpha,\beta)} \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha+\beta}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ where $\delta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and inc is the inclusion operator from $(\mathcal{H}, K^{\alpha+\beta})$ into $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$.*

We need the following lemmas for the generalization of Theorem 4.5.

Lemma 4.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and T be a bounded linear operator on $\mathcal{H}_1$. Then*

$$\ker(T \otimes I_{\mathcal{H}_2}) = \ker T \otimes \mathcal{H}_2.$$

Proof. It is easily seen that $\ker T \otimes \mathcal{H}_2 \subset \ker(T \otimes I_{\mathcal{H}_2})$. To establish the opposite inclusion, let x be an arbitrary element in $\ker(T \otimes I_{\mathcal{H}_2})$. Fix an orthonormal basis $\{f_i\}$ of $\mathcal{H}_2$. Note that x is of the form $\sum v_i \otimes f_i$ for some v_i's in $\mathcal{H}_1$. Since $x \in \ker(T \otimes I_{\mathcal{H}_2})$, we have $\sum Tv_i \otimes f_i = 0$. Moreover, since $\{f_i\}$ is an orthonormal basis of $\mathcal{H}_2$, it follows that $Tv_i = 0$ for all i. Hence x belongs to $\ker(T) \otimes \mathcal{H}_2$, completing the proof of the lemma. □

Lemma 4.9. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. If $B_1, \ldots, B_m$ are closed subspaces of $\mathcal{H}_1$, then*

$$\bigcap_{l=1}^{m}(B_l \otimes \mathcal{H}_2) = \Big(\bigcap_{l=1}^{m} B_l\Big) \otimes \mathcal{H}_2.$$

Proof. We only prove the non-trivial inclusion, namely, $\cap_{l=1}^m (B_l \otimes \mathcal{H}_2) \subset (\cap_{l=1}^m B_l) \otimes \mathcal{H}_2$.

Let $\{f_j\}_j$ be an orthonormal basis of $\mathcal{H}_2$ and x be an arbitrary element in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Recall that x can be written uniquely as $\sum x_j \otimes f_j$, $x_j \in \mathcal{H}_1$.

Claim: If x belongs to $B_l \otimes \mathcal{H}_2$, then x_j belongs to B_l for all j.

To prove the claim, assume that $\{e_i\}_i$ is an orthonormal basis of B_l. Since $\{e_i \otimes f_j\}_{i,j}$ is an orthonormal basis of $B_l \otimes \mathcal{H}_2$ and x can be written as $\sum x_{ij} e_i \otimes f_j = \sum_j (\sum_i x_{ij} e_i) \otimes f_j$. Then, the uniqueness of the representation $x = \sum x_j \otimes f_j$, ensures that $x_j = \sum_i x_{ij} e_i$. In particular, x_j belongs to B_l for all j. Thus the claim is verified.

Now let y be any element in $\cap_{l=1}^m (B_l \otimes \mathcal{H}_2)$. Let $\sum y_j \otimes f_j$ be the unique representation of y in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then from the claim, it follows that $y_j \in \cap_{l=1}^m B_l$. Consequently, $y \in (\cap_{l=1}^m B_l) \otimes \mathcal{H}_2$. This completes the proof. □

The proof of the following lemma is straightforward and therefore it is omitted.

Lemma 4.10. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ be a bounded linear operator and $B : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator. Then*

$$\ker BAB^* = B(\ker A).$$

The lemma given below is a generalization of [6, Lemma 1.22 (i)] to commuting tuples. Recall that for a commuting m-tuple $\boldsymbol{T} = (T_1, \ldots, T_m)$, the operator $\boldsymbol{T}^{\boldsymbol{i}}$ is defined by $T_1^{i_1} \cdots T_m^{i_m}$, where $\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$.

Lemma 4.11. *If $K : \Omega \times \Omega \to \mathbb{C}$ is a positive definite kernel such that the m-tuple of multiplication operators $\boldsymbol{M}_z = (M_{z_1}, \ldots, M_{z_m})$ on $(\mathcal{H}, K)$ is bounded, then for $w \in \Omega$ and $\boldsymbol{i} = (i_1, \ldots, i_m), \boldsymbol{j} = (j_1, \ldots, j_m)$ in $\mathbb{Z}_+^m$,*

(i) $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$ *if* $|\boldsymbol{i}| > |\boldsymbol{j}|$,
(ii) $(\mathbf{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = \boldsymbol{j}! \delta_{\boldsymbol{i}\boldsymbol{j}} K(\cdot, w)$ *if* $|\boldsymbol{i}| = |\boldsymbol{j}|$.

Proof. First, we claim that if $i_l > j_l$ for some $1 \leq l \leq m$, then $(M_{z_l}^* - \bar{w}_l)^{i_l} \bar{\partial}_l^{j_l} K(\cdot, w) = 0$. The claim is verified by induction on j_l. The case $j_l = 0$ holds trivially since $(M_{z_l}^* - \bar{w}_l) K(\cdot, w) = 0$. Now assume that the claim is valid for $j_l = p$. We have to show that it is true for $j_l = p+1$ also. Suppose $i_l > p+1$. Then $i_l - 1 > p$. Hence, by the induction hypothesis, $(M_{z_l}^* - \bar{w}_l)^{i_l - 1} \bar{\partial}_l^p K(\cdot, w) = 0$. Differentiating this with respect to $\bar{w}_l$, we see that

$$(i_l - 1)(M_{z_l}^* - \bar{w}_l)^{i_l - 2}(-1) \bar{\partial}_l^p K(\cdot, w) + (M_{z_l}^* - \bar{w}_l)^{i_l - 1} \bar{\partial}_l^{p+1} K(\cdot, w) = 0.$$

Applying $(M_{z_l}^* - \bar{w}_l)$ to both sides of the equation above, we obtain

$$(i_l - 1)(M_{z_l}^* - \bar{w}_l)^{i_l - 1}(-1) \bar{\partial}_l^p K(\cdot, w) + (M_{z_l}^* - \bar{w}_l)^{i_l} \bar{\partial}_l^{p+1} K(\cdot, w) = 0.$$

Using the induction hypothesis once again, we conclude that $(M_{z_l}^* - \bar{w}_l)^{i_l} \bar{\partial}_l^{p+1} K(\cdot, w) = 0$. Hence the claim is verified.

Now, to prove the first part of the lemma, assume that $|\boldsymbol{i}| > |\boldsymbol{j}|$. Then there exists a l such that $i_l > j_l$. Hence from the claim, we have $(M_{z_l}^* - \bar{w}_l)^{i_l} \bar{\partial}_l^{j_l} K(\cdot, w) = 0$. Differentiating with respect to all other variables except $\bar{w}_l$, we get $(M_{z_l}^* - \bar{w}_l)^{i_l} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$. Applying the operator $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i} - i_l e_l}$, where e_l is the lth standard unit vector of $\mathbb{C}^m$, we see that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$, completing the proof of the first part.

For the second part, assume that $|\boldsymbol{i}| = |\boldsymbol{j}|$ and $\boldsymbol{i} \neq \boldsymbol{j}$. Then there is atleast one l such that $i_l > j_l$. Hence by the argument used in the last paragraph, we conclude that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$. Finally, if $\boldsymbol{i} = \boldsymbol{j}$, we use induction on $\boldsymbol{i}$ to proof the lemma. There is nothing to prove if $\boldsymbol{i} = 0$. For the proof by induction, now, assume that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = \boldsymbol{i}! K(\cdot, w)$ for some $\boldsymbol{i} \in \mathbb{Z}_+^m$. To complete the induction step, we have to prove that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i} + e_l} \bar{\partial}^{\boldsymbol{i} + e_l} K(\cdot, w) = (\boldsymbol{i} + e_l)! K(\cdot, w)$. By the first part of the lemma, we have $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i} + e_l} \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = 0$. Differentiating with respect to $\bar{w}_l$, we get that

$$(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i} + e_l} \bar{\partial}^{\boldsymbol{i} + e_l} K(\cdot, w) - (i_l + 1)(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = 0.$$

Hence, by the induction hypothesis, $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i} + e_l} \bar{\partial}^{\boldsymbol{i} + e_l} K(\cdot, w) = (\boldsymbol{i} + e_l)! K(\cdot, w)$. This completes the proof. ∎

Corollary 4.12. *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a positive definite kernel. Suppose that the m-tuple of multiplication operators $\boldsymbol{M}_z$ on $(\mathcal{H}, K)$ is bounded. Then, for all $w \in \Omega$, the set $\{ \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m \}$ is linearly independent. Consequently, the matrix $\left(\partial^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(w, w)\right)_{\boldsymbol{i}, \boldsymbol{j} \in \Lambda}$ is positive definite for any finite subset Λ of $\mathbb{Z}_+^m$.*

Proof. Let w be an arbitrary point in Ω. It is enough to show that the set $\{ \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k \}$ is linearly independent for each non-negative integer k. Since K is positive definite, there is nothing to prove if $k = 0$. To complete the proof by induction on k, assume that the set $\{ \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k \}$ is linearly independent for some non-negative integer k. Suppose that $\sum_{|\boldsymbol{i}| \leq k+1} a_{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = 0$

for some $a_{\boldsymbol{i}}$'s in $\mathbb{C}$. Then $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{q}}(\sum_{|\boldsymbol{i}|\le k+1} a_{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}}K(\cdot,w)) = 0$, for all $\boldsymbol{q}\in\mathbb{Z}_+^m$ with $|\boldsymbol{q}|\le k+1$. If $|\boldsymbol{q}| = k+1$, by Lemma 4.11, we have that $a_{\boldsymbol{q}}\ q!K(\cdot,w) = 0$. Consequently, $a_{\boldsymbol{q}} = 0$ for all $\boldsymbol{q}\in\mathbb{Z}_+^m$ with $|\boldsymbol{q}| = k+1$. Hence, by the induction hypothesis, we conclude that $a_{\boldsymbol{i}} = 0$ for all $\boldsymbol{i}\in\mathbb{Z}_+^m$, $|\boldsymbol{i}|\le k+1$ and the set $\left\{\ \bar{\partial}^{\boldsymbol{i}}K(\cdot,w) : \boldsymbol{i}\in\mathbb{Z}_+^m, |\boldsymbol{i}|\le k+1\ \right\}$ is linearly independent, completing the proof of the first part of the corollary.

If Λ is a finite subset of $\mathbb{Z}_+^m$, then it follows form the linear independence of the vectors $\left\{\bar{\partial}^{\boldsymbol{i}}K(\cdot,w) : \boldsymbol{i}\in\Lambda\right\}$ that the matrix $\left(\left\langle\bar{\partial}^{\boldsymbol{j}}K(\cdot,w), \bar{\partial}^{\boldsymbol{i}}K(\cdot,w)\right\rangle\right)_{\boldsymbol{i},\boldsymbol{j}\in\Lambda}$ is positive definite. Now the proof is complete since $\left\langle\bar{\partial}^{\boldsymbol{j}}K(\cdot,w), \bar{\partial}^{\boldsymbol{i}}K(\cdot,w)\right\rangle = \partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}}K(w,w)$ (see Proposition 2.2). $\square$

The following proposition is also a generalization to the multi-variate setting of [6, Lemma 1.22 (ii)](see also [7]).

Proposition 4.13. *If $K:\Omega\times\Omega\to\mathbb{C}$ is a sharp kernel, then for every $w\in\Omega$*

$$\bigcap_{|\boldsymbol{j}|=k+1}\ker\,(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}} = \bigvee\left\{\bar{\partial}^{\boldsymbol{j}}K(\cdot,w) : |\boldsymbol{j}|\le k\right\}.$$

Proof. The inclusion $\bigvee\{\bar{\partial}^{\boldsymbol{j}}K(\cdot,w) : |\boldsymbol{j}|\le k\}\subseteq\bigcap_{|\boldsymbol{j}|=k+1}\ker\,(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}$ follows from part (i) of Lemma 4.11. We use induction on k for the opposite inclusion. From the definition of sharp kernel, this inclusion is evident if $k=0$. Assume that

$$\bigcap_{|\boldsymbol{j}|=k+1}\ker\,(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}} \subseteq \bigvee\left\{\bar{\partial}^{\boldsymbol{j}}K(\cdot,w) : |\boldsymbol{j}|\le k\right\}$$

for some non-negative integer k. To complete the proof by induction, we show that the inclusion remains valid for $k+1$ as well. Let f be an arbitrary element of $\bigcap_{|\boldsymbol{i}|=k+2}\ker(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}}$. Fix a $\boldsymbol{j}\in\mathbb{Z}_+^m$ with $|\boldsymbol{j}| = k+1$. Then it follows that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}f$ belongs to $\cap_{l=1}^m\ker(M_{z_l}^* - \bar{w}_l)$. Since K is sharp, we see that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}f = c_{\boldsymbol{j}}K(\cdot,w)$ for some constant $c_{\boldsymbol{j}}$ depending on w. Therefore

$$\begin{aligned}(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}\Big(f - \sum_{|\boldsymbol{q}|=k+1}\frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!}\bar{\partial}^{\boldsymbol{q}}K(\cdot,w)\Big) =& c_{\boldsymbol{j}}K(\cdot,w) - \sum_{|\boldsymbol{q}|=k+1}\frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!}(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}\bar{\partial}^{\boldsymbol{q}}K(\cdot,w)\\ =& c_{\boldsymbol{j}}K(\cdot,w) - \sum_{|\boldsymbol{q}|=k+1}c_{\boldsymbol{q}}\delta_{\boldsymbol{j}\boldsymbol{q}}\tfrac{\boldsymbol{j}!}{\boldsymbol{q}!}K(\cdot,w)\\ =&0,\end{aligned}$$

where the last equality follows from Lemma 4.11. Hence the element $f - \sum_{|\boldsymbol{q}|=k+1}\frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!}\bar{\partial}^{\boldsymbol{q}}K(\cdot,w)$ belongs to $\bigcap_{|\boldsymbol{j}|=k+1}\ker(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{j}}$. Thus by the induction hypothesis, $f - \sum_{|\boldsymbol{q}|=k+1}\frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!}\bar{\partial}^{\boldsymbol{q}}K(\cdot,w) = \sum_{|\boldsymbol{j}|\le k}d_{\boldsymbol{j}}\bar{\partial}^{\boldsymbol{j}}K(\cdot,w)$. Hence f belongs to $\bigvee\{\bar{\partial}^{\boldsymbol{j}}K(\cdot,w) : |\boldsymbol{j}|\le k+1\}$. This completes the proof. $\square$

For a m-tuple of bounded operators $\mathbf{T} = (T_1,\ldots,T_m)$ on a Hilbert space $\mathcal{H}$, we define an operator $D^{\mathbf{T}} : \mathcal{H}\bigoplus\cdots\bigoplus\mathcal{H}\to\mathcal{H}$ by

$$D^{\boldsymbol{T}}(x_1,\ldots,x_m) = \sum_{i=1}^m T_ix_i,\ \ x_1,\ldots,x_m\in\mathcal{H}.$$

A routine verification shows that $(D_{\boldsymbol{T}})^* = D^{\boldsymbol{T}^*}$. The following lemma is undoubtedly well known, however, we provide a proof for the sake of completeness.

Lemma 4.14. *Let $K:\Omega\times\Omega\to\mathbb{C}$ be a positive definite kernel such that the m-tuple of multiplication operators $\boldsymbol{M}_z$ on $(\mathcal{H},K)$ is bounded. Let $w=(w_1,\ldots,w_m)$ be a fixed but arbitrary point in Ω and let $\mathcal{V}_w$ be the subspace given by $\{f\in(\mathcal{H},K) : f(w)=0\}$. Then K is a generalized Bergman kernel if and only if for every $w\in\Omega$,*

$$\mathcal{V}_w = \Big\{\textstyle\sum_{i=1}^m(z_i-w_i)g_i : g_i\in(\mathcal{H},K)\Big\}. \tag{4.1}$$

Proof. First, observe that the right-hand side of (4.1) is equal to ran $D^{\boldsymbol{M}_z - w}$. Hence it suffices to show that K is a generalized Bergman kernel if and only if $\mathcal{V}_w = \text{ran } D^{\boldsymbol{M}_z - w}$. In any case, we have the following inclusions

$$\begin{aligned} \text{ran } D^{\boldsymbol{M}_z - w} = \text{ran } (D_{(\boldsymbol{M}_z - w)^*})^* \subseteq \overline{\text{ran } (D_{(\boldsymbol{M}_z - w)^*})^*} &= \ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} \\ &\subseteq \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp} \\ &= \mathcal{V}_w. \end{aligned} \tag{4.2}$$

Hence it follows that $\mathcal{V}_w = \text{ran } D^{\boldsymbol{M}_z - w}$ if and only if equality is forced everywhere in these inclusions, that is, ran $(D_{(\boldsymbol{M}_z - w)^*})^* = \overline{\text{ran } (D_{(\boldsymbol{M}_z - w)^*})^*}$ and $\ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} = \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp}$. Now ran $(D_{(\boldsymbol{M}_z - w)^*})^* = \overline{\text{ran } (D_{(\boldsymbol{M}_z - w)^*})^*}$ if and only if ran $(D_{(\boldsymbol{M}_z - w)^*})^*$ is closed. Recall that, if $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert spaces, and an operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ has closed range, then T^* also has closed range. Therefore, ran $(D_{(\boldsymbol{M}_z - w)^*})^*$ is closed if and only if ran $D_{(\boldsymbol{M}_z - w)^*}$ is closed. Finally, note that $\ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} = \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp}$ holds if and only if $\ker D_{(\boldsymbol{M}_z - w)^*} = \{cK(\cdot, w) : c \in \mathbb{C}\}$. This completes the proof. □

Notation 4.15. *Recall that for $1 \le i \le m$, $M_i^{(1)}, M_i^{(2)}, J_k M_i$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathcal{H}, K_1), (\mathcal{H}, K_2)$ and $(\mathcal{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta})$, respectively. Set $\boldsymbol{M}^{(1)} = (M_1^{(1)}, \ldots, M_m^{(1)})$, $\boldsymbol{M}^{(2)} = (M_1^{(2)}, \ldots, M_m^{(2)})$ and $\boldsymbol{J}_k\boldsymbol{M} = (J_k M_1, \ldots, J_k M_m)$. Also, for the sake of brevity, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces $(\mathcal{H}, K_1)$ and $(\mathcal{H}, K_2)$, respectively for the rest of this section.*

The following lemma is the main tool to prove that the kernel $J_k(K_1, K_2)_{|\text{res}\,\Delta}$ is sharp whenever K_1 and K_2 are sharp.

Lemma 4.16. *If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels, then for all $w = (w_1, \ldots, w_m) \in \Omega$,*

$$\begin{aligned} \bigcap_{p=1}^m \ker\Big(\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k^{\perp}}\Big) &= \bigcap_{|\boldsymbol{i}|=1} \ker\big(\boldsymbol{M}^{(1)} - w\big)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker\big(\boldsymbol{M}^{(2)} - w\big)^{*\boldsymbol{i}} \\ &= \bigvee\big\{K_1(\cdot, w) \otimes \bar{\partial}^{\boldsymbol{i}} K_2(\cdot, w) : |\boldsymbol{i}| \le k\big\}. \end{aligned}$$

Proof. Since K_1 and K_2 are sharp kernels, by Proposition 4.13, it follows that

$$\bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} = \bigvee\{K_1(\cdot, w) \otimes \bar{\partial}^{\boldsymbol{j}} K_2(\cdot, w) : |\boldsymbol{j}| \le k\}. \tag{4.3}$$

Therefore, if we can show that

$$\bigcap_{p=1}^m \ker\Big(\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k{}^{\perp}}\Big) = \bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}}, \tag{4.4}$$

then we will be done. To prove this, first note that

$$\begin{aligned} \bigcap_{p=1}^m \ker\Big(\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathcal{A}_k^{\perp}}\Big) &= \Big(\bigcap_{p=1}^m \ker\big((M_p^{(1)} - w_p)^* \otimes I\big)\Big) \bigcap \mathcal{A}_k^{\perp} \\ &= \Big(\bigcap_{p=1}^m \big(\ker(M_p^{(1)} - w_p)^* \otimes \mathcal{H}_2\big)\Big) \bigcap \mathcal{A}_k^{\perp} \\ &= \Big(\Big(\bigcap_{p=1}^m \ker(M_p^{(1)} - w_p)^*\Big) \otimes \mathcal{H}_2\Big) \bigcap \mathcal{A}_k^{\perp} \\ &= \Big(\ker D_{(\boldsymbol{M}^{(1)} - w)^*} \otimes \mathcal{H}_2\Big) \bigcap \mathcal{A}_k^{\perp}. \end{aligned}$$

Here the second equality follows from Lemma 4.8 and the third equality follows from Lemma 4.9. In view of the above computation, to verify (4.4), it is enough to show that

$$\Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes\mathcal{H}_2\Big)\bigcap\mathcal{A}_k^{\perp}=\bigcap_{|\boldsymbol{i}|=1}\ker{(\boldsymbol{M}^{(1)}-w)^*}^{\boldsymbol{i}}\otimes\bigcap_{|\boldsymbol{i}|=k+1}\ker{(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}. \tag{4.5}$$

Since K_1 is a sharp kernel, $\ker D_{(\boldsymbol{M}^{(1)}-w)^*}$ is spanned by the vector $K_1(\cdot,w)$. It is also easy to see that the vector $K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{j}}K_2(\cdot,w)$ belongs to $\mathcal{A}_k^{\perp}$ and hence, it is in $\Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes\mathcal{H}_2\Big)\bigcap\mathcal{A}_k^{\perp}$ for all $\boldsymbol{j}$ in $\mathbb{Z}_+^m$ with $|\boldsymbol{j}|\le k$. Therefore, by (4.3), we have the inclusion

$$\bigcap_{|\boldsymbol{i}|=1}\ker{(\boldsymbol{M}^{(1)}-w)^*}^{\boldsymbol{i}}\otimes\bigcap_{|\boldsymbol{i}|=k+1}\ker{(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}\subseteq\Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes\mathcal{H}_2\Big)\bigcap\mathcal{A}_k^{\perp}. \tag{4.6}$$

Now to prove the opposite inclusion, note that an arbitrary vector of $\big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes\mathcal{H}_2\big)\bigcap\mathcal{A}_k^{\perp}$ can be taken to be of the form $K_1(\cdot,w)\otimes g$, where $g\in\mathcal{H}_2$ is such that $K_1(\cdot,w)\otimes g\in\mathcal{A}_k^{\perp}$. We claim that such a vector g must be in $\bigcap_{|\boldsymbol{i}|=k+1}\ker{(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}$.

As before, we realize the vectors of $\mathcal{H}_1\otimes\mathcal{H}_2$ as functions in $z=(z_1,\ldots,z_m),\zeta=(\zeta_1,\ldots,\zeta_m)$ in Ω. Fix any $\boldsymbol{i}\in\mathbb{Z}_+^m$ with $|\boldsymbol{i}|=k+1$. Then $(\zeta-z)^{\boldsymbol{i}}=(\zeta_{q_1}-z_{q_1})(\zeta_{q_2}-z_{q_2})\cdots(\zeta_{q_{k+1}}-z_{q_{k+1}})$ for some $1\le q_1,q_2,\ldots,q_{k+1}\le m$. Since $M_i^{(1)}$ and $M_i^{(2)}$ are bounded for $1\le i\le m$, for any $h\in\mathcal{H}_1\otimes\mathcal{H}_2$, we see that the function $(\zeta-z)^{\boldsymbol{i}}h$ belongs to $\mathcal{H}_1\otimes\mathcal{H}_2$. Then

$$\begin{aligned}
&\big\langle K_1(\cdot,w)\otimes g,(\zeta_{q_1}-z_{q_1})(\zeta_{q_2}-z_{q_2})\cdots(\zeta_{q_{k+1}}-z_{q_{k+1}})h\big\rangle\\
&\quad=\Big\langle M^*_{(\zeta_{q_1}-z_{q_1})}(K_1(\cdot,w)\otimes g),(\zeta_{q_2}-z_{q_2})\cdots(\zeta_{q_{k+1}}-z_{q_{k+1}})h\Big\rangle\\
&\quad=\Big\langle (I\otimes {M_{q_1}^{(2)}}^*-{M_{q_1}^{(1)}}^*\otimes I)K_1(\cdot,w)\otimes g,(\zeta_{q_2}-z_{q_2})\cdots(\zeta_{q_{k+1}}-z_{q_{k+1}})h\Big\rangle\\
&\quad=\Big\langle K_1(\cdot,w)\otimes {M_{q_1}^{(2)}}^*g-\bar w_{q_1}K_1(\cdot,w)\otimes g,(\zeta_{q_2}-z_{q_2})\cdots(\zeta_{q_{k+1}}-z_{q_{k+1}})h\Big\rangle\\
&\quad=\Big\langle K_1(\cdot,w)\otimes (M_{q_1}^{(2)}-w_{q_1})^*g,(\zeta_{q_2}-z_{q_2})\cdots(\zeta_{q_{k+1}}-z_{q_{k+1}})h\Big\rangle.
\end{aligned}$$

Repeating this process, we get

$$\Big\langle K_1(\cdot,w)\otimes g,(\zeta-z)^{\boldsymbol{i}}h\Big\rangle=\Big\langle K_1(\cdot,w)\otimes{(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}g,h\Big\rangle.$$

Since $|\boldsymbol{i}|=k+1$, it follows that the element $(\zeta-z)^{\boldsymbol{i}}h$ belongs to $\mathcal{A}_k$. Furthermore, since $K_1(\cdot,w)\otimes g\in\mathcal{A}_k^{\perp}$, from the above equality, we have

$$\Big\langle K_1(\cdot,w)\otimes{(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}g,h\Big\rangle=0$$

for any $h\in\mathcal{H}_1\otimes\mathcal{H}_2$. Taking $h=K_1(\cdot,w)\otimes K_2(\cdot,u)$, $u\in\Omega$, we get $K_1(w,w)\big({(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}g\big)(u)=0$ for all $u\in\Omega$. Since $K_1(w,w)>0$, it follows that ${(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}g=0$. Since this is true for all $\boldsymbol{i}\in\mathbb{Z}_+^m$ with $|\boldsymbol{i}|=k+1$, it follows that $g\in\bigcap_{|\boldsymbol{i}|=k+1}\ker{(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}}$. Hence $K_1(\cdot,w)\otimes g$ belongs to

$$\bigcap_{|\boldsymbol{i}|=1}\ker{(\boldsymbol{M}^{(1)}-w)^*}^{\boldsymbol{i}}\otimes\bigcap_{|\boldsymbol{i}|=k+1}\ker{(\boldsymbol{M}^{(2)}-w)^*}^{\boldsymbol{i}},$$

proving the opposite inclusion of (4.6). This completes the proof of equality in (4.4). ☐

Theorem 4.17. *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain. If $K_1,K_2:\Omega\times\Omega\to\mathbb{C}$ are two sharp kernels, then so is the kernel $J_k(K_1,K_2)_{|\operatorname{res}\Delta}$, $k\ge 0$.*

Proof. Since the tuple $\boldsymbol{M}^{(1)}$ is bounded, by Corollary 4.6, it follows that the tuple $\boldsymbol{J}_k\boldsymbol{M}$ is also bounded. Now we will show that the kernel $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}$ is positive definite on $\Omega\times\Omega$. Since K_2 is positive definite, by Corollary 4.12, we obtain that the matrix $\big(\partial^{\boldsymbol{i}}\bar\partial^{\boldsymbol{j}}K_2(w,w)\big)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^{k}$ is positive definite for $w\in\Omega$. Moreover, since K_1 is also positive definite, we conclude that $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(w,w)$ is positive definite for $w\in\Omega$. Hence, by [8, Lemma 3.6], we conclude that the kernel $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}$ is positive definite.

To complete the proof, we need to show that

$$\ker D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*}=\mathrm{ran}\ J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(\cdot,w),\ w\in\Omega.$$

Note that, by the definition of R and J_k (see the discussion before Theorem 1.3), we have

$$RJ_k(K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{i}}K_2(\cdot,w))=J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(\cdot,w)e_{\boldsymbol{i}},\ \boldsymbol{i}\in\mathbb{Z}_+^m,\ |\boldsymbol{i}|\le k. \tag{4.7}$$

In the computation below, the third equality follows from Lemma 4.10, the injectivity of the map $RJ_{k|\mathcal{A}_k^\perp}$ implies the fourth equality, the fifth equality follows from Lemma 4.16 and finally the last equality follows from (4.7):

$$\begin{aligned}\ker D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*}&=\bigcap_{p=1}^m\ker(J_kM_p-w_p)^*\\&=\bigcap_{p=1}^m\ker\Big((RJ_k)P_{\mathcal{A}_k^\perp}\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathcal{A}_k^\perp}(RJ_k)^*\Big)\\&=\bigcap_{p=1}^m(RJ_k)\Big(\ker\big(P_{\mathcal{A}_k^\perp}\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathcal{A}_k^\perp}\big)\Big)\\&=(RJ_k)\Big(\bigcap_{p=1}^m\ker\big(P_{\mathcal{A}_k^\perp}\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathcal{A}_k^\perp}\big)\Big)\\&=(RJ_k)\Big(\bigvee\big\{\,K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{i}}K_2(\cdot,w):|\boldsymbol{j}|\le k\big\}\Big)\\&=\mathrm{ran}\ J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(\cdot,w).\end{aligned}$$

This completes the proof. □

The lemma given below is the main tool to prove Theorem 4.19.

Lemma 4.18. *Let $K_1,K_2:\Omega\times\Omega\to\mathbb{C}$ be two generalized Bergman kernels, and let $w=(w_1,\dots,w_m)$ be an arbitrary point in Ω. Suppose that f is a function in $\mathcal{H}_1\otimes\mathcal{H}_2$ satisfying $\big(\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f(z,\zeta)\big)_{|z=\zeta=w}=0$ for all $\boldsymbol{i}\in\mathbb{Z}_+^m,\ |\boldsymbol{i}|\le k$. Then*

$$f(z,\zeta)=\sum_{j=1}^m(z_j-w_j)f_j(z,\zeta)+\sum_{|\boldsymbol{q}|=k+1}(z-\zeta)^{\boldsymbol{q}}f_{\boldsymbol{q}}^{\sharp}(z,\zeta)$$

for some functions $f_j,f_{\boldsymbol{q}}^{\sharp}$ in $\mathcal{H}_1\otimes\mathcal{H}_2, j=1,\dots,m,\ \boldsymbol{q}\in\mathbb{Z}_+^m,|\boldsymbol{q}|=k+1$.

Proof. Since K_1 and K_2 are generalized Bergman kernels, by Theorem 4.5, we have that $K_1\otimes K_2$ is also a generalized Bergman kernel. Therefore, if f is a function in $\mathcal{H}_1\otimes\mathcal{H}_2$ vanishing at (w,w), then using Lemma 4.14, we find functions $f_1,\dots,f_m$, and $g_1,\dots,g_m$ in $\mathcal{H}_1\otimes\mathcal{H}_2$ such that

$$f(z,\zeta)=\sum_{j=1}^m(z_j-w_j)f_j+\sum_{j=1}^m(\zeta_j-w_j)g_j.$$

Equivalently, we have

$$f(z,\zeta) = \sum_{j=1}^{m}(z_j - w_j)(f_j + g_j) + \sum_{j=1}^{m}(z_j - \zeta_j)(-g_j).$$

Thus the statement of the lemma is verified for $k=0$. To complete the proof by induction on k, assume that the statement is valid for some non-negative integer k. Let f be a function in $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that $\big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z,\zeta)\big)_{|z=\zeta=w} = 0$ for all $\boldsymbol{i} \in \mathbb{Z}_+^m,\ |\boldsymbol{i}| \leq k+1$. By induction hypothesis, we can write

$$f(z,\zeta) = \sum_{j=1}^{m}(z_j - w_j) f_j(z,\zeta) + \sum_{|\boldsymbol{q}|=k+1} (z-\zeta)^{\boldsymbol{q}} f^{\sharp}_{\boldsymbol{q}}(z,\zeta) \tag{4.8}$$

for some $f_j, f^{\sharp}_{\boldsymbol{q}} \in \mathcal{H}_1 \otimes \mathcal{H}_2,\ j=1,\ldots,m,\ \boldsymbol{q} \in \mathbb{Z}_+^m,\ |\boldsymbol{q}| = k+1$. Fix a $\boldsymbol{i} \in \mathbb{Z}_+^m$ with $|\boldsymbol{i}| = k+1$. Applying $\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}}$ to both sides of (4.8), we see that

$$\begin{aligned}
\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z,\zeta) &= \sum_{j=1}^{m}(z_j - w_j)\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f_j(z,\zeta) + \sum_{|\boldsymbol{q}|=k+1} \big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}}\big((z-\zeta)^{\boldsymbol{q}} f^{\sharp}_{\boldsymbol{q}}(z,\zeta)\big)\\
&= \sum_{j=1}^{m}(z_j - w_j)\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f_j(z,\zeta) + \sum_{|\boldsymbol{q}|=k+1}\sum_{\boldsymbol{p}\leq \boldsymbol{i}} \tbinom{\boldsymbol{i}}{\boldsymbol{p}}\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{p}}(z-\zeta)^{\boldsymbol{q}}\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}-\boldsymbol{p}} f^{\sharp}_{\boldsymbol{q}}(z,\zeta).
\end{aligned}$$

Putting $z=\zeta=w$, we obtain

$$\big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z,\zeta)\big)_{|z=\zeta=w} = (-1)^{|\boldsymbol{i}|}\boldsymbol{i}!\ f^{\sharp}_{\boldsymbol{i}}(w,w),$$

where we have used the simple identity: $\Big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{p}}(z-\zeta)^{\boldsymbol{q}}\Big)_{|z=\zeta=w} = \delta_{\boldsymbol{p}\boldsymbol{q}}(-1)^{|\boldsymbol{p}|}p!$.

Since $\big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z,\zeta)\big)_{|z=\zeta=w} = 0$, we conclude that $f^{\sharp}_{\boldsymbol{i}}(w,w) = 0$. Since the statement of the lemma has been shown to be valid for $k=0$, it follows that

$$f^{\sharp}_{\boldsymbol{i}}(z,\zeta) = \sum_{j=1}^{m}(z_j - w_j)\big(f^{\sharp}_{\boldsymbol{i}}\big)_j(z,\zeta) + \sum_{j=1}^{m}(z_j - \zeta_j)\big(f^{\sharp}_{\boldsymbol{i}}\big)^{\sharp}_j(z,\zeta) \tag{4.9}$$

for some $\big(f^{\sharp}_{\boldsymbol{i}}\big)_j,\ \big(f^{\sharp}_{\boldsymbol{i}}\big)^{\sharp}_j \in \mathcal{H}_1 \otimes \mathcal{H}_2,\ j=1,\ldots,m$. Since (4.9) is valid for any $\boldsymbol{i} \in \mathbb{Z}_+^m,\ |\boldsymbol{i}| = k+1$, replacing the $f^{\sharp}_{\boldsymbol{q}}$'s in (4.8) by $\sum_{j=1}^{m}(z_j - w_j)\big(f^{\sharp}_{\boldsymbol{q}}\big)_j(z,\zeta) + \sum_{j=1}^{m}(z_j - \zeta_j)\big(f^{\sharp}_{\boldsymbol{q}}\big)^{\sharp}_j(z,\zeta)$, we obtain the desired conclusion after some straightforward algebraic manipulation. □

Theorem 4.19. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are generalized Bergman kernels, then so is the kernel $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}$, $k \geq 0$.*

Proof. By Theorem 4.17, we will be done if we can show that ran $D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*}$ is closed for every $w \in \Omega$. Fix a point $w = (w_1,\ldots,w_m)$ in Ω. Let $\boldsymbol{X} := \big(P_{\mathcal{A}_k^{\perp}}(M_1^{(1)} \otimes I)_{|\mathcal{A}_k^{\perp}},\ldots,P_{\mathcal{A}_k^{\perp}}(M_m^{(1)} \otimes I)_{|\mathcal{A}_k^{\perp}}\big)$. By Corollary 4.6, we see that ran $D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*}$ is closed if and only if ran $D_{(\boldsymbol{X}-w)^*}$ is closed. Moreover, since $(D_{(\boldsymbol{X}-w)^*})^* = D^{(\boldsymbol{X}-w)}$, we conclude that ran $D_{(\boldsymbol{X}-w)^*}$ is closed if and only if ran $D^{(\boldsymbol{X}-w)}$ is closed. Note that X satisfies the following equality:

$$\ker D_{(\boldsymbol{X}-w)^*}{}^{\perp} = \overline{\mathrm{ran}\ (D_{(\boldsymbol{X}-w)^*})^*} = \overline{\mathrm{ran}\ D^{(\boldsymbol{X}-w)}}.$$

Therefore, to prove ran $D^{(\boldsymbol{X}-w)}$ is closed, it is enough to show that $\ker D_{(\boldsymbol{X}-w)^*}{}^{\perp} \subseteq$ ran $D^{\boldsymbol{X}-w}$. To prove this, note that

$$D^{(\boldsymbol{X}-w)}(g_1 \oplus \cdots \oplus g_m) = P_{\mathcal{A}_k^{\perp}}\big(\sum_{i=1}^{m}(z_i - w_i)g_i\big),\ g_i \in \mathcal{A}_k^{\perp}, i=1,\ldots,m.$$

Thus

$$\text{ran } D^{(\boldsymbol{X}-w)} = \Big\{ P_{\mathcal{A}_k^\perp}\big(\sum_{i=1}^m (z_i - w_i) g_i : g_1, \ldots, g_m \in \mathcal{A}_k^\perp \Big\}. \tag{4.10}$$

Now, let f be an arbitrary element of $\ker D_{(\boldsymbol{X}-w)^*}{}^\perp$. Then, by Lemma 4.16 and Proposition 2.2, we have $\big((\frac{\partial}{\partial\zeta})^{\boldsymbol{i}} f(z,\zeta)\big)_{|z=\zeta=w} = 0$ for all $\boldsymbol{i} \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| \leq k$. By Lemma 4.18,

$$f(z,\zeta) = \sum_{j=1}^m (z_j - w_j) f_j(z,\zeta) + \sum_{|\boldsymbol{q}|=k+1} (z-\zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^\sharp(z,\zeta)$$

for some functions $f_j, f_{\boldsymbol{q}}^\sharp$ in $\mathcal{H}_1 \otimes \mathcal{H}_2, j = 1, \ldots, m$ and $\boldsymbol{q} \in \mathbb{Z}_+^m, |\boldsymbol{q}| = k+1$. Note that the element $\sum_{|\boldsymbol{q}|=k+1} (z-\zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^\sharp$ belongs to $\mathcal{A}_k$. Hence $f = P_{\mathcal{A}_k^\perp}(f) = P_{\mathcal{A}_k^\perp}\big(\sum_{j=1}^m (z_j - w_j) f_j\big)$. Furthermore, since the subspace $\mathcal{A}_k$ is invariant under $(M_j^{(1)} - w_j)$, $j = 1, \ldots, m$, we see that

$$\begin{aligned} f = P_{\mathcal{A}_k^\perp}\big(\textstyle\sum_{j=1}^m (z_j - w_j) f_j\big) &= P_{\mathcal{A}_k^\perp}\Big(\textstyle\sum_{j=1}^m (z_j - w_j)\big(P_{\mathcal{A}_k^\perp} f_j + P_{\mathcal{A}_k} f_j\big)\Big) \\ &= P_{\mathcal{A}_k^\perp}\big(\textstyle\sum_{j=1}^m (z_j - w_j)(P_{\mathcal{A}_k^\perp} f_j)\big). \end{aligned}$$

Therefore, from (4.10), we conclude that $f \in \text{ran } D^{(\boldsymbol{X}-w)}$. This completes the proof. □

4.1. **The class $\mathcal{F}B_2(\Omega)$.** In this subsection, first we will use Theorem 4.19 to prove that, if $\Omega \subset \mathbb{C}$, and K^α, K^β, defined on $\Omega \times \Omega$, are generalized Bergman kernels, then so is the kernel $\mathbb{K}^{(\alpha,\beta)}$. The following proposition, which is interesting on its own right, is an essential tool in proving this theorem. The notation below is chosen to be close to that of [16].

Proposition 4.20. *Let $\Omega \subset \mathbb{C}$ be a bounded domain. Let T be a bounded linear operator of the form $\begin{bmatrix} T_0 & S \\ 0 & T_1 \end{bmatrix}$ on $H_0 \bigoplus H_1$. Suppose that T belongs to $B_2(\Omega)$ and T_0 belongs to $B_1(\Omega)$. Then T_1 belongs to $B_1(\Omega)$.*

Proof. First, note that, for $w \in \Omega$,

$$(T - w)(x \oplus y) = ((T_0 - w)x + Sy) \oplus (T_1 - w)y. \tag{4.11}$$

Since $T \in B_2(\mathbb{D})$, $T - w$ is onto. Hence, from the above equality, it follows that $(T_1 - w)$ is onto.

Now we claim that $\dim\ker(T_1 - w) = 1$ for all $w \in \Omega$. From (4.11), we see that $(x \oplus y)$ belongs to $\ker(T - w)$ if and only if $(T_0 - w)x + Sy = 0$ and $y \in \ker(T_1 - w)$. Therefore, if $\dim\ker(T_1 - w)$ is 0, it must follow that $\ker(T - w) = \ker(T_0 - w)$, which is a contradiction. Hence $\dim\ker(T_1 - w)$ is atleast 1. Now assume that $\dim\ker(T_1 - w) > 1$. Let $v_1(w)$ and $v_2(w)$ be two linearly independent vectors in $\ker(T_1 - w)$. Since $(T_0 - w)$ is onto, there exist $u_1(w), u_2(w) \in H_0$ such that $(T_0 - w)u_i(w) + Sv_i(w) = 0$, $i = 1, 2$. Hence the vectors $(u_1(w) \oplus v_1(w)), (u_2(w) \oplus v_2(w))$ belong to $\ker(T - w)$. Also, since $\dim\ker(T_0 - w) = 1$, there exists $\gamma(w) \in H_0$, such that $(\gamma(w) \oplus 0)$ belongs to $\ker(T - w)$. It is easy to verify that the vectors $\{(u_1(w) \oplus v_1(w)), (u_2(w) \oplus v_2(w)), (\gamma(w) \oplus 0)\}$ are linearly independent. This is a contradiction since $\dim\ker(T - w) = 2$. Therefore $\dim\ker(T_1 - w) \leq 1$. In consequence, $\dim\ker(T_1 - w) = 1$.

Finally, to show that $\bigvee_{w\in\Omega} \ker(T_1 - w) = H_1$, let y be an arbitrary vector in H_1 which is orthogonal to $\bigvee_{w\in\Omega} \ker(T_1 - w)$. Then it follows that $(0 \oplus y)$ is orthogonal to $\ker(T - w), w \in \Omega$. Consequently, $y = 0$. This completes the proof. □

Theorem 4.21. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are positive definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that the operators $M^{(\alpha)^*}$ on $(\mathcal{H}, K^\alpha)$ and $M^{(\beta)^*}$ on $(\mathcal{H}, K^\beta)$ belong to $B_1(\Omega^*)$. Then the operator $\mathbb{M}^{(\alpha,\beta)^*}$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. Equivalently, if K^α and K^β are generalized Bergman kernels, then so is the kernel $\mathbb{K}^{(\alpha,\beta)}$.*

Proof. Since the operators ${M^{(\alpha)}}^*$ and ${M^{(\beta)}}^*$ belong to $B_1(\Omega^*)$, it follows from Theorem 4.19 that the kernel $J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta}$ is a generalized Bergman kernel. Therefore, from corollary 4.7, we deduce that the operator $\begin{pmatrix} {M^{(\alpha+\beta)}}^* & \eta\, \mathsf{inc}^* \\ 0 & {\mathbb{M}^{(\alpha,\beta)}}^* \end{pmatrix}$ belongs to $B_2(\Omega^*)$, where $\eta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and inc is the inclusion operator from $(\mathcal{H}, K^{\alpha+\beta})$ into $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. Also, by Theorem 4.5, the operator ${M^{(\alpha+\beta)}}^*$ on $(\mathcal{H}, K^{\alpha+\beta})$ belongs to $B_1(\Omega^*)$. Proposition 4.20, therefore shows that the operator ${\mathbb{M}^{(\alpha,\beta)}}^*$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. ☐

A smaller class of operators $\mathcal{F}B_n(\Omega)$ from $B_n(\Omega)$, $n \geq 2$, was introduced in [16]. A set of tractable complete unitary invariants and concrete models were given for operators in this class. We give below examples of a large class of operators in $\mathcal{F}B_2(\Omega)$. In case Ω is the unit disc $\mathbb{D}$, these examples include the homogeneous operators of rank 2 in $B_2(\mathbb{D})$ which are known to be in $\mathcal{F}B_2(\mathbb{D})$.

Definition 4.22. *An operator T on $H_0 \bigoplus H_1$ is said to be in $\mathcal{F}B_2(\Omega)$ if it is of the form $\begin{bmatrix} T_0 & S \\ 0 & T_1 \end{bmatrix}$, where $T_0, T_1 \in B_1(\Omega)$ and S is a non-zero operator satisfying $T_0 S = S T_1$.*

Theorem 4.23. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are positive definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that the operators ${M^{(\alpha)}}^*$ on $(\mathcal{H}, K^\alpha)$ and ${M^{(\beta)}}^*$ on $(\mathcal{H}, K^\beta)$ belong to $B_1(\Omega^*)$. Then the operator $(J_1 M)^*$ on $(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta})$ belongs to $\mathcal{F}B_2(\Omega^*)$.*

Proof. By Theorem 4.19, the operator $(J_1 M)^*$ on $(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta})$ belongs to $B_2(\Omega^*)$, and by Corollary 4.7, it is unitarily equivalent to $\begin{pmatrix} {M^{(\alpha+\beta)}}^* & \eta\, \mathsf{inc}^* \\ 0 & {\mathbb{M}^{(\alpha,\beta)}}^* \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha+\beta}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$. By Theorem 4.5, the operator ${M^{(\alpha+\beta)}}^*$ on $(\mathcal{H}, K^{\alpha+\beta})$ belongs to $B_1(\Omega^*)$ and by Theorem 4.21, the operator ${\mathbb{M}^{(\alpha,\beta)}}^*$ on $(\mathcal{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. The adjoint of the inclusion operator inc clearly intertwines ${M^{(\alpha+\beta)}}^*$ and ${\mathbb{M}^{(\alpha,\beta)}}^*$. Therefore the operator $(J_1 M)^*$ on $(\mathcal{H}, J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta})$ belongs to $\mathcal{F}B_2(\Omega^*)$. ☐

Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions $K^{\alpha_1}, K^{\alpha_2}, K^{\beta_1}$ and K^{β_2} are positive definite on $\Omega \times \Omega$ for some $\alpha_i, \beta_i > 0$, $i = 1, 2$. Suppose that the operators ${M^{(\alpha_i)}}^*$ on $(\mathcal{H}, K^{\alpha_i})$ and ${M^{(\beta_i)}}^*$ on $(\mathcal{H}, K^{\beta_i})$, $i = 1, 2$, belong to $B_1(\Omega^*)$. Let $\mathcal{A}_1(\alpha_i, \beta_i)$ be the subspace $\mathcal{A}_1$ of the Hilbert space $(\mathcal{H}, K^{\alpha_i}) \otimes (\mathcal{H}, K^{\beta_i})$ for $i = 1, 2$. Then we have the following corollary.

Corollary 4.24. *The operators $\big(M^{(\alpha_1)} \otimes I\big)^*_{|\mathcal{A}_1(\alpha_1,\beta_1)^\perp}$ and $\big(M^{(\alpha_2)} \otimes I\big)^*_{|\mathcal{A}_1(\alpha_2,\beta_2)^\perp}$ are unitarily equivalent if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

Proof. If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then there is nothing to prove. For the converse, assume that the operators $\big(M^{(\alpha_1)} \otimes I\big)^*_{|\mathcal{A}_1(\alpha_1,\beta_1)^\perp}$ and $\big(M^{(\alpha_2)} \otimes I\big)^*_{|\mathcal{A}_1(\alpha_2,\beta_2)^\perp}$ are unitarily equivalent. Then, by Corollary 3.10, we see that the operators $\begin{pmatrix} {M^{(\alpha_1+\beta_1)}}^* & \eta_1\, (\mathsf{inc})_1^* \\ 0 & {\mathbb{M}^{(\alpha_1,\beta_1)}}^* \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha_1+\beta_1}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha_1,\beta_1)})$ and $\begin{pmatrix} {M^{(\alpha_2+\beta_2)}}^* & \eta_2\, (\mathsf{inc})_2^* \\ 0 & {\mathbb{M}^{(\alpha_2,\beta_2)}}^* \end{pmatrix}$ on $(\mathcal{H}, K^{\alpha_2+\beta_2}) \bigoplus (\mathcal{H}, \mathbb{K}^{(\alpha_2,\beta_2)})$ are unitarily equivalent, where $\eta_i = \frac{\beta_i}{\sqrt{\alpha_i\beta_i(\alpha_i+\beta_i)}}$ and $(\mathsf{inc})_i$ is the inclusion operator from $(\mathcal{H}, K^{\alpha_i+\beta_i})$ into $(\mathcal{H}, \mathbb{K}^{(\alpha_i,\beta_i)})$, $i = 1, 2$. Since ${M^{(\alpha_i)}}^*$ on $(\mathcal{H}, K^{\alpha_i})$ and ${M^{(\beta_i)}}^*$ on $(\mathcal{H}, K^{\beta_i})$, $i = 1, 2$, belong to $B_1(\Omega^*)$, by Theorem 4.23, we conclude that the operator $\begin{pmatrix} {M^{(\alpha_i+\beta_i)}}^* & \eta_i\, (\mathsf{inc})_i^* \\ 0 & {\mathbb{M}^{(\alpha_i,\beta_i)}}^* \end{pmatrix}$ belongs to $\mathcal{F}B_2(\Omega^*)$ for $i = 1, 2$. Therefore, by [16, Theorem 2.10], we obtain that

$$\mathcal{K}_{{M^{(\alpha_1+\beta_1)}}^*} = \mathcal{K}_{{M^{(\alpha_2+\beta_2)}}^*} \quad \text{and} \quad \frac{\eta_1\, \|(\mathsf{inc})_1^*(t_1)\|^2}{\|t_1\|^2} = \frac{\eta_2\, \|(\mathsf{inc})_2^*(t_2)\|^2}{\|t_2\|^2}, \tag{4.12}$$

where $\mathcal{K}_{{M^{(\alpha_i+\beta_i)}}^*}$, $i = 1, 2$, is the curvature of the operator ${M^{(\alpha_i+\beta_i)}}^*$, and t_1 and t_2 are two non-vanishing holomorphic sections of the vector bundles $E_{{\mathbb{M}^{(\alpha_1,\beta_1)}}^*}$ and $E_{{\mathbb{M}^{(\alpha_2,\beta_2)}}^*}$, respectively. Note that, for $i = 1, 2$, $t_i(w) = \mathbb{K}^{(\alpha_i,\beta_i)}(\cdot, w)$ is a holomorphic non-vanishing section of the vector bundle

$E_{\mathbb{M}^{(\alpha_i,\beta_i)*}}$, and also $(\mathsf{inc})_i^*(\mathbb{K}^{(\alpha_i,\beta_i)}(\cdot,w)) = K^{\alpha_i+\beta_i}(\cdot,w)$, $w \in \Omega$. Therefore the second equality in (4.12) implies that

$$\frac{\eta_1 K^{\alpha_1+\beta_1}(w,w)}{K^{\alpha_1+\beta_1}(w,w)\partial\bar{\partial}\log K(w,w)} = \frac{\eta_2 K^{\alpha_2+\beta_2}(w,w)}{K^{\alpha_2+\beta_2}(w,w)\partial\bar{\partial}\log K(w,w)}, \quad w \in \Omega,$$

or equivalently $\eta_1 = \eta_2$. Furthermore, it is easy to see that $\mathcal{K}_{M^{(\alpha_1+\beta_1)*}} = \mathcal{K}_{M^{(\alpha_2+\beta_2)*}}$ if and only if $\alpha_1+\beta_1 = \alpha_2+\beta_2$. Hence, from (4.12), we see that

$$\alpha_1+\beta_1 = \alpha_2+\beta_2 \quad \text{and} \quad \eta_1 = \eta_2. \tag{4.13}$$

Then a simple calculation shows that (4.13) is equivalent to $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, completing the proof. □

5. The generalized Wallach set

Let Ω be a bounded domain in $\mathbb{C}^m$. Recall that the Bergman space $A^2(\Omega)$ is the Hilbert space of all square integrable analytic functions defined on Ω. The inner product of $A^2(\Omega)$ is given by the formula

$$\langle f, g\rangle := \int_\Omega f(z)\overline{g(z)}\ \mathrm{dV(z)},\ \ \mathrm{f, g} \in \mathrm{A}^2(\Omega),$$

where dV(z) is the area measure on $\mathbb{C}^m$. The evaluation linear functional $f \mapsto f(w)$ is bounded on $A^2(\Omega)$ for all $w \in \Omega$. Consequently, the Bergman space is a reproducing kernel Hilbert space. The reproducing kernel of the Bergman space $A^2(\Omega)$ is called the Bergman kernel of Ω and is denoted by B_Ω.

If $\Omega \subset \mathbb{C}^m$ is a bounded symmetric domain, then the ordinary Wallach set $\mathcal{W}_\Omega$ is defined as $\{t > 0 : B_\Omega^t \text{ is non-negative definite}\}$. Here B_Ω^t, $t > 0$, makes sense since every bounded symmetric domain Ω is simply connected and the Bergman kernel on it is non-vanishing. If Ω is the Euclidean unit ball $\mathbb{B}_m$, then the Bergman kernel is given by

$$B_{\mathbb{B}_m}(z,w) = (1-\langle z,w\rangle)^{-(m+1)},\ \ z,w \in B_{\mathbb{B}_m}, \tag{5.1}$$

and the Wallach set $\mathcal{W}_{\mathbb{B}_m} = \{t \in \mathbb{R} : t > 0\}$. But, in general, there are examples of bounded symmetric domains, like the open unit ball in the space of all $m \times n$ matrices, $m, n > 1$, with respect to the operator norm, where the Wallach set is a proper subset of $\{t \in \mathbb{R} : t > 0\}$. An explicit description of the Wallach set $\mathcal{W}_\Omega$ for a bounded symmetric domain Ω is given in [12].

Replacing the Bergman kernel in the definition of the Wallach set by an arbitrary scalar valued non-negative definite kernel K, we define the ordinary Wallach set $\mathcal{W}(K)$ to be the set

$$\{t > 0 : K^t \text{ is non-negative definite}\}.$$

Here we have assumed that there exists a continuous branch of logarithm of K on $\Omega\times\Omega$ and therefore K^t, $t > 0$, makes sense. Clearly, every natural number belongs to the Wallach set $\mathcal{W}(K)$. In [4], it is shown that K^t is non-negative definite for all $t > 0$ if and only if $\big(\partial_i\bar{\partial}_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite. Therefore it follows from the discussion in the previous paragraph that there are non-negative definite kernels K on $\Omega\times\Omega$ for which $\big(\partial_i\bar{\partial}_j \log K(z,w)\big)_{i,j=1}^m$ need not define a non-negative definite kernel on $\Omega\times\Omega$. However, it follows from Proposition 2.3 that $K^{t_1+t_2}\big(\partial_i\bar{\partial}_j \log K(z,w)\big)_{i,j=1}^m$ is a non-negative kernel on $\Omega\times\Omega$ as soon as t_1 and t_2 are in the Wallach set $\mathcal{W}(K)$. Therefore it is natural to introduce the generalized Wallach set for any scalar valued kernel K defined on $\Omega\times\Omega$ as follows:

$$G\mathcal{W}(K) := \big\{t \in \mathbb{R} :\ K^{t-2}\mathbb{K} \text{ is non-negative definite}\big\}, \tag{5.2}$$

where, as before, we have assumed that K^t is well defined for all $t \in \mathbb{R}$. Clearly, we have the following inclusion

$$\big\{t_1+t_2\ :\ t_1,t_2 \in \mathcal{W}(K)\big\} \subseteq G\mathcal{W}(K).$$

5.1. **Generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$.** In this section, we compute the generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$. In the case of the unit disc $\mathbb{D}$, the Bergman kernel $B_{\mathbb{D}}(z,w) = (1 - z\bar{w})^{-2}$ and $\partial\bar{\partial}\log B_{\mathbb{D}}(z,w) = 2(1-z\bar{w})^{-2}$, $z, w \in \mathbb{D}$. Therefore t is in $G\mathcal{W}(B_{\mathbb{D}})$ if and only if $(1-z\bar{w})^{-(2t+2)}$ is non-negative definite on $\mathbb{D} \times \mathbb{D}$. Consequently, $G\mathcal{W}(B_{\mathbb{D}}) = \{t \in \mathbb{R} : t \geq -1\}$. For the case of the Bergman kernel $B_{\mathbb{B}_m}$ of the Euclidean unit ball $\mathbb{B}_m$, $m \geq 2$, we have shown that $G\mathcal{W}(B_{\mathbb{B}_m}) = \{t \in \mathbb{R} : t \geq 0\}$. The proof is obtained by putting together a number of lemmas which are of independent interest.

Before computing the generalized Wallach set $G\mathcal{W}(B_{\mathbb{B}_m})$ for the Bergman kernel of the Euclidean ball $\mathbb{B}_m$, we point out that the result is already included in [23, Theorem 3.7], see also [19, 15]. The justification for our detailed proofs in this particular case is that it is direct and elementary in nature.

As before, we write $K \succeq 0$ to denote that K is a non-negative definite kernel. For two non-negative definite kernels $K_1, K_2 : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$, we write $K_1 \preceq K_2$ if $K_2 - K_1$ is a non-negative definite kernel on $\Omega \times \Omega$. Analogously, we write $K_1 \succeq K_2$ if $K_1 - K_2$ is non-negative definite.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{C}^m$, and $\lambda_0 > 0$ be an arbitrary constant. Let $\{K_\lambda\}_{\lambda \geq \lambda_0}$ be a family of non-negative definite kernels, defined on $\Omega \times \Omega$, taking values in $\mathcal{M}_k(\mathbb{C})$ such that*

(i) *if $\lambda \geq \lambda' \geq \lambda_0$, then $K_{\lambda'} \preceq K_\lambda$,*
(ii) *for $z, w \in \Omega$, $K_\lambda(z,w)$ converges to $K_{\lambda_0}(z,w)$ entrywise as $\lambda \to \lambda_0$.*

Any $f : \Omega \to \mathbb{C}^k$ which is holomorphic and is in $(\mathcal{H}, K_\lambda)$ for all $\lambda > \lambda_0$ belongs to $(\mathcal{H}, K_{\lambda_0})$ if and only if $\sup_{\lambda > \lambda_0} \|f\|_{(\mathcal{H}, K_\lambda)} < \infty$.

Proof. Recall that if K and K' are two non-negative definite kernels satisfying $K \preceq K'$, then $(\mathcal{H}, K) \subseteq (\mathcal{H}, K')$ and $\|h\|_{(\mathcal{H},K')} \leq \|h\|_{(\mathcal{H},K)}$ for $h \in (\mathcal{H}, K)$ (see [24, Theorem 6.25]). Therefore, by the hypothesis, we have that

$$(\mathcal{H}, K_{\lambda'}) \subseteq (\mathcal{H}, K_\lambda) \quad \text{and} \quad \|h\|_{(\mathcal{H},K_\lambda)} \leq \|h\|_{(\mathcal{H},K_{\lambda'})}, \tag{5.3}$$

whenever $\lambda \geq \lambda' \geq \lambda_0$ and $h \in (\mathcal{H}, K_{\lambda'})$.

Now assume that $f \in (\mathcal{H}, K_{\lambda_0})$. Then, clearly $\|f\|_{(\mathcal{H},K_\lambda)} \leq \|f\|_{(\mathcal{H},K_{\lambda_0})}$ for all $\lambda > \lambda_0$. Consequently, $\sup_{\lambda>\lambda_0} \|f\|_{(\mathcal{H},K_\lambda)} \leq \|f\|_{(\mathcal{H},K_{\lambda_0})} < \infty$. For the converse, assume that $\sup_{\lambda>\lambda_0} \|f\|_{(\mathcal{H},K_\lambda)} < \infty$. Then, from (5.3), it follows that $\lim_{\lambda\to\lambda_0} \|f\|_{(\mathcal{H},K_\lambda)}$ exists and is equal to $\sup_{\lambda>\lambda_0} \|f\|_{(\mathcal{H},K_\lambda)}$. Since $f \in (\mathcal{H}, K_\lambda)$ for all $\lambda > \lambda_0$, by [24, Theorem 6.23], we have that

$$f(z)f(w)^* \preceq \|f\|^2_{(\mathcal{H},K_\lambda)} K_\lambda(z,w).$$

Taking limit as $\lambda \to \lambda_0$ and using part (ii) of the hypothesis, we obtain

$$f(z)f(w)^* \preceq \sup_{\lambda>\lambda_0} \|f\|^2_{(\mathcal{H},K_\lambda)} K_{\lambda_0}(z,w).$$

Hence, using [24, Theorem 6.23] once again, we conclude that $f \in (\mathcal{H}, K_{\lambda_0})$. ∎

If $m \geq 2$, then from (5.1), we have

$$\begin{aligned}&\Big(\big(B^t_{\mathbb{B}_m} \partial_i \bar{\partial}_j \log B_{\mathbb{B}_m}\big)(z,w)\Big)_{i,j=1}^m \\ &= \frac{m+1}{(1 - \langle z, w\rangle)^{t(m+1)+2}} \begin{pmatrix} 1 - \sum_{j\neq 1} z_j\bar{w}_j & z_2\bar{w}_1 & \cdots & z_m\bar{w}_1 \\ z_1\bar{w}_2 & 1 - \sum_{j\neq 2} z_j\bar{w}_j & \cdots & z_m\bar{w}_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1\bar{w}_m & z_2\bar{w}_m & \cdots & 1 - \sum_{j\neq m} z_j\bar{w}_j \end{pmatrix}.\end{aligned} \tag{5.4}$$

For $m \geq 2$, $\lambda \in \mathbb{R}$ and $z, w \in \mathbb{B}_m$, set

$$\mathbb{K}_\lambda(z,w) := \frac{1}{(1-\langle z,w\rangle)^\lambda}\begin{pmatrix} 1-\sum_{j\neq 1} z_j\bar{w}_j & z_2\bar{w}_1 & \cdots & z_m\bar{w}_1 \\ z_1\bar{w}_2 & 1-\sum_{j\neq 2} z_j\bar{w}_j & \cdots & z_m\bar{w}_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1\bar{w}_m & z_2\bar{w}_m & \cdots & 1-\sum_{j\neq m} z_j\bar{w}_j \end{pmatrix}. \tag{5.5}$$

In view (5.4) and (5.5), for $\lambda > 2$, we have

$$\mathbb{K}_\lambda = \frac{2}{t(m+1)}\Big(\, (B_{\mathbb{B}_m}^{\frac{t}{2}})^2 \partial_i\bar{\partial}_j \log B_{\mathbb{B}_m}^{\frac{t}{2}}\,\Big)_{i,j=1}^m,$$

where $t = \frac{\lambda-2}{m+1} > 0$. Since $B_{\mathbb{B}_m}^{t/2}$ is positive definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $t > 0$, it follows from Corollary 2.4 that $\mathbb{K}_\lambda$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $\lambda > 2$. Since $\mathbb{K}_\lambda(z,w) \to \mathbb{K}_2(z,w)$, $z, w \in \mathbb{B}_m$, entrywise as $\lambda \to 2$, we conclude that $\mathbb{K}_2$ is also non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$.

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{C}^m$. The lemma given below finds the norm of the vector $z_2 \otimes e_1$ in $(\mathcal{H}, \mathbb{K}_\lambda)$ when $\lambda > 2$.

Lemma 5.2. *For each $\lambda > 2$, the vector $z_2 \otimes e_1$ belongs to $(\mathcal{H}, \mathbb{K}_\lambda)$ and $\|z_2 \otimes e_1\|_{(\mathcal{H},\mathbb{K}_\lambda)} = \sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$.*

Proof. By a straight forward computation, we obtain

$$\bar{\partial}_1 \mathbb{K}_\lambda(\cdot, 0)e_2 = z_2 \otimes e_1 + (\lambda-1) z_1 \otimes e_2$$

and

$$\bar{\partial}_2 \mathbb{K}_\lambda(\cdot, 0)e_1 = (\lambda-1) z_2 \otimes e_1 + z_1 \otimes e_2.$$

Thus we have

$$(\lambda-1)\bar{\partial}_2 \mathbb{K}_\lambda(\cdot,0)e_1 - \bar{\partial}_1 \mathbb{K}_\lambda(\cdot,0)e_2 = (\lambda^2 - 2\lambda) z_2 \otimes e_1. \tag{5.6}$$

By Proposition 2.2, the vectors $\bar{\partial}_2\mathbb{K}_\lambda(\cdot,0)e_1$ and $\bar{\partial}_1\mathbb{K}_\lambda(\cdot,0)e_2$ belong to $(\mathcal{H}, \mathbb{K}_\lambda)$. Since $\lambda > 2$, from (5.6), it follows that the vector $z_2 \otimes e_1$ belongs to $(\mathcal{H}, \mathbb{K}_\lambda)$. Now, taking norm in both sides of (5.6) and using Proposition 2.2 a second time, we obtain

$$\begin{aligned} &(\lambda^2-2\lambda)^2 \|z_2 \otimes e_1\|^2 \\ &= (\lambda-1)^2 \langle \partial_2\bar{\partial}_2 \mathbb{K}_\lambda(0,0)e_1, e_1\rangle - (\lambda-1)\langle \partial_1\bar{\partial}_2\mathbb{K}_\lambda(0,0)e_1, e_2\rangle \\ &\quad - (\lambda-1)\langle \bar{\partial}_1\partial_2 \mathbb{K}_\lambda(0,0)e_2, e_1\rangle + \langle \partial_1\bar{\partial}_1\mathbb{K}_\lambda(0,0)e_2, e_2\rangle \end{aligned} \tag{5.7}$$

By a routine computation, we obtain

$$\partial_i\bar{\partial}_j \mathbb{K}_\lambda(0,0) = (\lambda-1)\delta_{ij} I_m + E_{ji},$$

where δ_{ij} is the Kronecker delta function, I_m is the identity matrix of order m, and E_{ji} is the matrix whose (j,i)th entry is 1 and all other entries are 0. Hence, from (5.7), we see that

$$\begin{aligned} &(\lambda^2-2\lambda)^2 ||z_2 \otimes e_1||^2 \\ &= (\lambda-1)^2(\lambda-1) - 2(\lambda-1) + (\lambda-1) \\ &= (\lambda-1)(\lambda^2-2\lambda). \end{aligned}$$

Hence $||z_2 \otimes e_1|| = \sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$, completing the proof of the lemma. □

Lemma 5.3. *The multiplication operator by the coordinate function z_2 on $(\mathcal{H}, \mathbb{K}_2)$ is not bounded.*

Proof. Since $\mathbb{K}_2(\cdot, 0)e_1 = e_1$, we have that the constant function e_1 is in $(\mathcal{H}, \mathbb{K}_2)$. Hence, to prove that M_{z_2} is not bounded on $(\mathcal{H}, \mathbb{K}_2)$, it suffices to show that the vector $z_2 \otimes e_1$ does not belong to $(\mathcal{H}, \mathbb{K}_2)$.

Consider the family of non-negative definite kernels $\{\mathbb{K}_\lambda\}_{\lambda \geq 2}$. Observe that for $\lambda \geq \lambda' \geq 2$,

$$\mathbb{K}_\lambda(z,w) - \mathbb{K}_{\lambda'}(z,w) = \Big((1-\langle z,w\rangle)^{-(\lambda-\lambda')} - 1\Big)\mathbb{K}_{\lambda'}(z,w). \tag{5.8}$$

It is easy to see that if $\lambda \geq \lambda'$, then $(1-\langle z,w\rangle)^{-(\lambda-\lambda')}-1 \succeq 0$. Thus the right hand side of (5.8), being a product of a scalar valued non-negative definite kernel with a matrix valued non-negative definite kernel, is non-negative definite. Consequently, $K_{\lambda'} \preceq K_\lambda$. Also since $\mathbb{K}_\lambda(z,w) \to \mathbb{K}_2(z,w)$ entry-wise as $\lambda \to 2$, by Lemma 5.1, it follows that $z_2 \otimes e_1 \in (\mathcal{H}, \mathbb{K}_2)$ if and only if $\sup_{\lambda>2} \|z_2 \otimes e_1\|_{(\mathcal{H},\mathbb{K}_\lambda)} < \infty$. By lemma 5.2, we have $\|z_2 \otimes e_1\|_{(\mathcal{H},\mathbb{K}_\lambda)} = \sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$. Thus $\sup_{\lambda>2} \|z_2 \otimes e_1\|_{(\mathcal{H},\mathbb{K}_\lambda)} = \infty$. Hence the vector $z_2 \otimes e_1$ does not belong to $(\mathcal{H}, \mathbb{K}_2)$ and the operator M_{z_2} on $(\mathcal{H}, \mathbb{K}_\lambda)$ is not bounded. □

The following theorem describes the generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$, $m \geq 2$.

Theorem 5.4. *If $m \geq 2$, then $G\mathcal{W}(B_{\mathbb{B}_m}) = \{t \in \mathbb{R} : t \geq 0\}$.*

Proof. In view of (5.4) and (5.5), we see that $t \in G\mathcal{W}(B_{\mathbb{B}_m})$ if and only if $\mathbb{K}_{t(m+1)+2}$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$. Hence we will be done if we can show that $\mathbb{K}_\lambda$ is non-negative if and only if $\lambda \geq 2$.

From the discussion preceding Lemma 5.2, we have that $\mathbb{K}_\lambda$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $\lambda \geq 2$.

To prove the converse, assume that $\mathbb{K}_\lambda$ is non-negative definite for some $\lambda < 2$. Note that $\mathbb{K}_2$ can be written as the product

$$\mathbb{K}_2(z,w) = (1-\langle z,w\rangle)^{-(2-\lambda)}\mathbb{K}_\lambda(z,w),\; z,w \in \mathbb{B}_m. \tag{5.9}$$

Also, the multiplication operator M_{z_2} on $(\mathcal{H}, (1-\langle z,w\rangle)^{-(2-\lambda)})$ is bounded. Hence, by Lemma 2.7, there exists a constant $c > 0$ such that $(c^2 - z_2\bar{w}_2)(1-\langle z,w\rangle)^{-(2-\lambda)}$ is non-negative definite. Consequently, the product $(c^2 - z_2\bar{w}_2)(1-\langle z,w\rangle)^{-(2-\lambda)}\mathbb{K}_\lambda$, which is $(c^2 - z_2\bar{w}_2)\mathbb{K}_2$, is non-negative. Hence, again by Lemma 2.7, it follows that the operator M_{z_2} is bounded on $(\mathcal{H}, \mathbb{K}_2)$. This is a contradiction to the Lemma 5.3. Hence our assumption that $\mathbb{K}_\lambda$ is non-negative for some $\lambda < 2$, is not valid. This completes the proof. □

6. Quasi-invariant kernels

In this section, we show that if K a is quasi-invariant kernel with respect to some J, then $K^{t-2}\mathbb{K}$ is also a quasi-invariant kernel with respect to $\mathbb{J} := J(\varphi,z)^t D\varphi(z)^{\mathrm{tr}}$, $\varphi \in \mathrm{Aut}(\Omega)$, $z \in \Omega$, whenever t is in the generalized Wallach set $G\mathcal{W}(K)$. The lemma given below, which will be used in the proof of the Proposition 6.2, follows from applying the chain rule [25, page 8] twice.

Lemma 6.1. *Let $\phi = (\phi_1, \ldots, \phi_m) : \Omega \to \mathbb{C}^m$ be a holomorphic map and $g : \mathrm{ran}\,\phi \to \mathbb{C}$ be a real analytic function. If $h = g \circ \phi$, then*

$$\Big(\big(\partial_i\bar{\partial}_j h\,\big)(z)\Big)_{i,j=1}^m = (D\phi(z))^{\mathrm{tr}}\Big(\big(\partial_i\bar{\partial}_j g\,\big)(\varphi(z))\Big)_{i,j=1}^m\overline{(D\phi(z))},$$

where $(D\phi)(z)^{\mathrm{tr}}$ is the transpose of the derivative of ϕ at z.

Proposition 6.2. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. Let $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel and $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathbb{C}\setminus\{0\}$ be a function such that $J(\varphi,\cdot)$ is holomorphic for each φ in $\mathrm{Aut}(\Omega)$. Suppose that K is quasi-invariant with respect to J. Then the kernel $K^{t-2}\mathbb{K}$ is also quasi-invariant with respect to $\mathbb{J}$ whenever $t \in G\mathcal{W}_\Omega(K)$, where $\mathbb{J}(\varphi,z) = J(\varphi,z)^t D\varphi(z)^{\mathrm{tr}}$, $\varphi \in \mathrm{Aut}(\Omega)$, $z \in \Omega$.*

Proof. Since K is quasi-invariant with respect to J, we have

$$\log K(z,z) = \log|J(\varphi,z)|^2 + \log K(\varphi(z),\varphi(z)),\; \varphi \in \mathrm{Aut}(\Omega),\; z \in \Omega.$$

Also, $J(\varphi,\cdot)$ is a non-vanishing holomorphic function on Ω, therefore $\partial_i\bar{\partial}_j \log|J(\varphi,z)|^2 = 0$. Hence

$$\partial_i\bar{\partial}_j \log K(z,z) = \partial_i\bar{\partial}_j \log K(\varphi(z),\varphi(z)),\; \varphi \in \mathrm{Aut}(\Omega),\; z \in \Omega. \tag{6.1}$$

Any biholomorphic automorphism φ of Ω is of the form $(\varphi_1, \ldots, \varphi_m)$, where $\varphi_i : \Omega \to \mathbb{C}$ is holomorphic, $i = 1, \ldots, m$. By setting $g(z) = \log K(z, z),\ z \in \Omega$, and using Lemma 6.1, we obtain

$$\big(\partial_i\bar\partial_j \log K(\varphi(z), \varphi(z))\big)_{i,j=1}^m = D\varphi(z)^{\rm tr}\big(\big(\partial_l\bar\partial_p \log K\big)(\varphi(z), \varphi(z))\big)_{l,p=1}^m \overline{D\varphi(z)}.$$

Combining this with (6.1), we obtain

$$\big(\partial_i\bar\partial_j \log K(z, z)\big)_{i,j=1}^m = D\varphi(z)^{\rm tr}\big(\big(\partial_l\bar\partial_p \log K\big)(\varphi(z), \varphi(z))\big)_{l,p=1}^m \overline{D\varphi(z)}. \tag{6.2}$$

Multiplying $K(z, z)^t$ both sides and using the quasi-invariance of K, a second time, we obtain

$$\begin{aligned}&\big(\ K(z, z)^t\partial_i\bar\partial_j \log K(z, z)\ \big)_{i,j=1}^m \\ &\quad = J(\varphi, z)^t D\varphi(z)^{\rm tr} K(\varphi(z), \varphi(z))^t \big(\big(\partial_l\bar\partial_p \log K\big)(\varphi(z), \varphi(z)\ \big)_{l,p=1}^m \overline{J(\varphi, z)^t D\varphi(z)}.\end{aligned}$$

Equivalently, we have

$$K^{t-2}(z, z)\mathbb{K}(z, z) = \mathbb{J}(\varphi, z)K^{t-2}(\varphi(z), \varphi(z))\mathbb{K}(\varphi(z), \varphi(z))\mathbb{J}(\varphi, z)^*, \tag{6.3}$$

where $\mathbb{J}(\varphi, z) = J(\varphi, z)^t D\varphi(z)^{\rm tr}$, $\varphi \in {\rm Aut}(\Omega),\ z \in \Omega$. Therefore, polarizing both sides of the above equation, we have the desired conclusion. □

Remark 6.3. *The function J in the definition of quasi-invariant kernel is said to be a projective cocycle if it is a Borel map satisfying*

$$J(\varphi\psi, z) = m(\varphi, \psi)J(\psi, z)J(\varphi, \psi z),\ \ \varphi, \psi \in {\rm Aut}(\Omega), z \in \Omega, \tag{6.4}$$

where $m : {\rm Aut}(\Omega) \times {\rm Aut}(\Omega) \to \mathbb{T}$ is a multiplier, that is, m is Borel and satisfies the following properties:

(i) $m(e, \varphi) = m(\varphi, e) = 1$, *where* $\varphi \in {\rm Aut}(\Omega)$ *and e is the identity in* ${\rm Aut}(\Omega)$
(ii) $m(\varphi_1, \varphi_2)m(\varphi_1\varphi_2, \varphi_3) = m(\varphi_1, \varphi_2\varphi_3)m(\varphi_2, \varphi_3)$, $\varphi_1, \varphi_2, \varphi_3 \in {\rm Aut}(\Omega)$.

J is said to be a cocycle if it is a projective cocycle with $m(\varphi, \psi) = 1$ for all φ, ψ in ${\rm Aut}(\Omega)$.

If $J : {\rm Aut}(\Omega) \times \Omega \to \mathbb{C} \setminus \{0\}$ in the Proposition 6.2 is a cocycle, then it is verified that the function $\mathbb{J}$ is a projective co-cycle. Moreover, if t is a positive integer, then $\mathbb{J}$ is also a cocycle.

For the preceding to be useful, one must exhibit non-negative definite kernels which are quasi-invariant. It is known that the Bergman kernel B_Ω of any bounded domain Ω is quasi-invariant with respect to J, where $J(\varphi, z) = \det D\varphi(z)$, $\varphi \in {\rm Aut}(\Omega), {\rm z} \in \Omega$.

Lemma 6.4. ([18, Proposition 1.4.12]) *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $\varphi : \Omega \to \Omega$ be a biholomorphic map. Then*

$$B_\Omega(z, w) = \det D\varphi(z)B_\Omega(\varphi(z), \varphi(w))\overline{\det D\varphi(w)},\ \ z, w \in \Omega.$$

The following proposition follows from combining Proposition 6.2 and Lemma 6.4, and therefore the proof is omitted.

Proposition 6.5. *Let Ω be a bounded domain $\mathbb{C}^m$. If t is in $GW(B_\Omega)$, then the kernel*

$$\boldsymbol{B}_\Omega^{(t)}(z, w) := \big(\ B_\Omega^t(z, w)\partial_i\bar\partial_j \log B_\Omega(z, w)\ \big)_{i,j=1}$$

is quasi-invariant with respect to $(\det D\varphi(z))^t D\varphi(z)^{\rm tr}$, $\varphi \in {\rm Aut}(\Omega),\ {\rm z} \in \Omega$.

For a fixed but arbitrary $\varphi \in {\rm Aut}(\Omega)$, let U_φ be the linear map on ${\rm Hol}(\Omega, \mathbb{C}^k)$ defined by

$$U_\varphi(f) = J\big(\varphi^{-1}, \cdot\big)f \circ \varphi^{-1},\ \ f \in {\rm Hol}(\Omega, \mathbb{C}^k). \tag{6.5}$$

The following proposition is a basic tool in defining unitary representations of the automorphism group ${\rm Aut}(\Omega)$. The straightforward proof for the case of unit disc $\mathbb{D}$ appears in [17]. The proof for the general domain Ω follows in exactly the same way.

Proposition 6.6. *The linear map U_φ is unitary on $(\mathcal{H}, K)$ for all φ in* ${\rm Aut}(\Omega)$ *if and only if the kernel K is quasi-invariant with respect to J.*

Let $Q:\Omega\to\mathcal{M}_k(\mathbb{C})$ be a real analytic function such that $Q(w)$ is positive definite for $w\in\Omega$. Let $\mathcal{H}$ be the Hilbert space of $\mathbb{C}^k$ valued holomorphic functions on Ω which are square integrable with respect to $Q(w)dV(w)$, that is,

$$\mathcal{H}=\big\{f\in \mathrm{Hol}(\Omega,\mathbb{C}^k):\|f\|^2:=\int_\Omega\langle Q(w)f(w),f(w)\rangle_{\mathbb{C}^k}dV(w)<\infty\big\},$$

where dV is the normalized volume measure on $\mathbb{C}^m$. Assume that the constant functions are in $\mathcal{H}$. The operator U_φ, defined in (6.5) is unitary if and only if

$$\begin{aligned}\|U_\varphi f\|^2 &= \int_\Omega\langle Q(w)(U_\varphi f)(w),(U_\varphi f)(w)\rangle dV(w)\\ &= \int_\Omega\langle \overline{J(\varphi^{-1},w)}^{\mathrm{tr}}Q(w)J(\varphi^{-1},w)f(\varphi^{-1}(w)),f(\varphi^{-1}(w))\rangle dV(w)\\ &= \int_\Omega\langle Q(w)f(w),f(w)\rangle dV(w),\end{aligned}$$

that is, if and only if Q transforms according to the rule

$$\overline{J(\varphi^{-1},w)}^{\mathrm{tr}}Q(w)J(\varphi^{-1},w)=Q(\varphi^{-1}(w))|\det(D\varphi^{-1})(w)|^2. \tag{6.6}$$

Set $J(\varphi^{-1},w)=\det(D\varphi^{-1}(w))^tD\varphi^{-1}(w)^{\mathrm{tr}}$ and $Q^{(t)}(w):=B_\Omega(w,w)^{1-t}\mathcal{K}(w,w)^{-1}$, where $\mathcal{K}(z,w):=\big(\partial_i\bar\partial_j\log B_\Omega(z,w)\big)_{i,j=1}^m$, $t>0$. Then $Q^{(t)}$ transforms according to the rule (6.6) since $\mathcal{K}$ transforms according to (6.2) and B_Ω transfomrs as in Lemma 6.4. If for some $t>0$, the Hilbert space $L^2_{\mathrm{hol}}(\Omega,Q^{(t)}\,dV)$ determined by the measure is nontrivial, then the corresponding reproducing kernel is of the form $B^t_\Omega(z,w)\mathcal{K}(z,w)$.

Let Ω be a bounded symmetric domain in $\mathbb{C}^m$. Note that if $K:\Omega\times\Omega\to\mathcal{M}_k(\mathbb{C})$ is a quasi-invariant kernel with respect to some J and the commuting tuple $\boldsymbol{M}_z=(M_{z_1},\ldots,M_{z_m})$ on $(\mathcal{H},K)$ is bounded, then the commuting tuple $\boldsymbol{M}_\varphi:=(M_{\varphi_1},\ldots,M_{\varphi_m})$ is unitarily equivalent to $\boldsymbol{M}_z$ via the unitary map U_φ, where $\varphi=(\varphi_1,\ldots,\varphi_m)$ is in $\mathrm{Aut}(\Omega)$. If t is in $G\mathcal{W}(B_\Omega)$ and the operator of multiplication M_{z_i} by the coordinate function z_i is bounded on the Hilbert space $(\mathcal{H},B_\Omega^{t/2})$, then it follows from Corollary 2.9 that the operator M_{z_i} on the Hilbert space $\big(\mathcal{H},\boldsymbol{B}_\Omega^{(t)}\big)$ is bounded as well. Therefore, in the language of [22], we conclude that the multiplication tuple $\boldsymbol{M}_z$ on $(\mathcal{H},\boldsymbol{B}_\Omega^{(t)})$ is homogeneous with respect to the group $\mathrm{Aut}(\Omega)$. In particular, if Ω is the Euclidean unit ball in $\mathbb{C}^m$, and t is any positive real number, then the multiplication tuple $\boldsymbol{M}_z$ on $(\mathcal{H},B_{\mathbb{B}_m}^{t/2})$ is bounded. Also, from Theorem 5.4, it follows that $\boldsymbol{B}_{\mathbb{B}_m}^{(t)}$ is non-negative definite. Consequently, the commuting m - tuple of operators $\boldsymbol{M}_z$ must be homogeneous with respect to the group $\mathrm{Aut}(\mathbb{B}_m)$.

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