Comments about some mathematical achievements by Adam Korányi

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of Adam Korányi

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Adam Korányi is born and studied in Szeged (Ungarn). He has been a student of B. Sz-Nagy, who is knowm for his work in Functional Analysis.

There was the communist time. Soviet Union controlled everything in Ungarn. On 23 October 1956, there was an uprising in Budapest, and the Soviet Army restaured the control on the country on 4 November with a violent repression.

Adam Korányi, with his friend L. Pukansky, left Ungarn in January 1957. After some month in western Europe, he went to the United States, and studied further in Chicago.

1957-1959
University of Chicago
Adam Korányi wrote a PhD Thesis under the supervision of M. H. Stone.

1960 - 1965 Then he visited the universities of Harvard, Berkeley, and Princeton.

1965 - 1979,
Yeshiva University, New-York

1979 - 1985,
Washington University, Saint-Louis

1985 - Distinguished Professor at Lehman College, and at the Graduate Center of the City University of New-York.

Loewner's Theorem

The first work by Adam Korányi is about Loewner's Theorem.
1955 - 1957

On the set of selfadjoint operators one considers a partial order:
$A \leq B$ if $B - A$ is a positive selfadjoint operator.
Real functions in one variable act on selfadjoint operators by functional calculus.

Question : Which functions preserve this order ?

Part of Loewner's Theorem says that a function which preserves this order is a Pick function: a holomorphic function which maps the upper halfplane into itself.

Theorems of Nevanlinna and Pick are related to Loewner's Theorem.
Adam Korányi's result is a contribution in this framework.

PhD Thesis under the supervision of M. H. Stone.

1957 - 1959

Generalization of his previous results in several variables:

Operator monotone functions in several variables, and a generalization of the Theorem of Nevanlinna - Pick.

1963

One paper with L. Pukansky: Holomorphic functions with positive real part on polycylinders.

Cayley transform

Visiting the universities of Harvard, Princeton, and Berkeley, Adam Korányi met S. Helgason and J. Wolf.

As a result he turned to the subject of Riemannian symmetric spaces, and especially to bounded symmetric domains.

Recall that the Cayley transform maps the unit disc in $\mathbb{C}$ onto the upperhalfplane:

$$w \mapsto z = i\frac{1+w}{1-w}.$$

Observation by Adam Korányi:

The Cayley transform corresponds to a $\frac{\pi}{2}$-rotation of the Riemann sphere. The stereographic projection interlaces this rotation and the Cayley transform.

With J. Wolf, A. Korányi has shown that the Cayley transform could be generalized to any bounded symmetric domain.

This is one of his major mathematical achievements.

Two papers:

1965 A realization of Hermitian symmetric spaces as generalized half-planes. In Annals of Math.

1965 Generalized Cayley transforms of bounded symmetric domains. In American Journal of Mathematics

In the simplex case the Cayley transform maps a bounded symmetric domain D to a tube of the form

$$V + i\Omega$$

where Ω is a symmetric cone in the vector space V. In this case one says that D is a domain of tube type.

In general the Cayley transform maps the bounded symmetric domain to a Siegel domain.

The book by Piatetskii-Shapiro considers the Cayley transform for classical domains.
The general case with the geometric meaning of the Cayley transform is due to A. Korányi and J. Wolf.

For a bounded symmetric domain $D = G/K$, the Cayley transform corresponds to an isometric transformation of the dual compact symmetric space $S = U/K$, where U is a compact real form of $G_{\mathbb{C}}$.

In the case of the unit disc D,

$$\begin{aligned} D &= SU(1,1)/SO(2) \\ G_{\mathbb{C}} &= SL(2,\mathbb{C}) \\ U &= SU(2) \\ S &= SU(2)/SO(2) \end{aligned}$$

Fatou Theorem

Let u be a harmonic function on the unit disc D, given as a Poisson integral on the unit circle B

$$u(z) = \int_B \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, d\mu(\zeta)$$

$$z = re^{i\theta},\ \zeta = e^{i\varphi}$$

Then, almost everywhere with respect to Lebesgue measure, the function $u(z)$ has boundary values

$$\lim_{z\to\zeta} = f^*(\zeta),$$

with the condition: z stays in the angular domain $S(\zeta,\varepsilon)$,

$$S(\zeta,\varepsilon) = \{z = re^{i\theta} \mid |\mathrm{angle}(\zeta - z, \zeta)| < \varepsilon\} \quad (0 < \epsilon < \frac{\pi}{2}).$$

For the general case of a bounded symmetric domain, A. Korányi found the right generalization of the non tangential limit. The problem was that a direct generalization of the non tangential convergence was not a notion invariant by the automorphism group of the domain. The main contribution was to find the right notion: admissible convergence.

With S. Helgason
1968 Fatou type Theorem for harmonic functions on symmetric spaces

With E. Stein
1968 Fatou's Theorem for Generalized Half-Planes

1969 Boundary Behavior of Harmonic Functions on Symmetric Spaces

Hua equations
The Laplace operator on the unit disc D for the Poincaré metric:

$$L = (1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

In case of the domain

$$\begin{aligned} D &= \{Z \in M(n, \mathbb{C}) \mid I - ZZ^* \text{ positive definite}\} \\ &= SU(n,n)/S\big(U(n) \times U(n)\big). \end{aligned}$$

Hua defines the operator valued differential operator

$$H = (I - ZZ^*)\partial_Z (I - Z^* Z)\partial_{Z^*}.$$

The Shilov boundary of the domain D is the unitary group $U(n)$. Hua shows that a Poisson-Shilov integral

$$F(Z) = \int_{U(n)} P(Z, U) d\mu(U),$$

satisfies the Hua system $HF = 0$

E. Stein raised the question:

Does the Hua system characterizes the Poisson-Shilov integrals ?

1975 First result by A. Korányi and P. Malliavin: for a domain of rank 2, by using the Brownian motion.

1980 A. Korányi and K. Johnson, for tube domains.

The general case is due to N. Berline and M. Vergne. In the non-tube case, one considers a third order system.

Some applications of Gelfand pairs in classical analysis, 1982

The following fact observed by A. Korányi has played an important role for harmonic analysts.

Consider the following pair

$$G_0 = U(n) \ltimes H_n,$$

the semidirect product of the unitary group $U(n)$ with the Heisenberg group $H_n \simeq \mathbb{R} \times \mathbb{C}^n$, and $K_0 = U(n)$.
The pair (G_0, K_0) is a Gelfand pair, or, equivalently, $L^1(H_n)^{K_0}$ is commutative.

More generally.

G/K Riemannian symmetric space of non compact type.
$G = MAN$ Iwasawa decomposition, A is a Cartan subgroup, N is nilpotent.
M centralizer of A in K.

A. Korányi has shown: If G/K is of rank one, the pair (MN, M) is a Gelfand pair.

Furthermore the spherical functions are given in terms of Laguerre polynomials.

This result has been the starting point of numerous mathematical works.

Quasi-conformal mappings

For a mapping $z \to w = f(z)$ $(z, w \in \mathbb{C})$, one defines the dilatation of f at z as

$$D_f(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}]}.$$

The mapping is conformal if the dilatation $D_f \equiv 1$.
The mapping is said to be quasi-conformal if the dilatation D_f is bounded.

With M. Reimann, A. Korányi studied conformal mappings on the nilpotent Heisenberg group operating on the boundary of complex hyperbolic n-spaces and transitive in the complement of one point.

1985 Appications quasi conformes sur le groupe de Heisenberg.

Analysis on symmetric cones and bounded symmetric domains

For a long time A. Korányi has been interested in the connection between the geometry of symmetric domains and Jordan algebras.
H. Upmeier will speak about it in next talk.

As A. Korányi was visiting Strasbourg around 1985, we decided to write a book on this topic. There were already several texts on the subject: Braun-Köcher, Rapaport, Satake,... Our aim was to write a self-contained text, and to developp the analysis by using the language of Jordan algebras. There were few new results, but there were new proofs, as the one for the computation of the Gindikin gamma function, and series expansions in terms of spherical polynomials.
At the same time we established a series expansion for the reproducing kernel of the weighed Bergman spaces on a bounded symmetric domain. This gave a new simple proof for the determination of the so-called Wallach set.

Unified approach to symmetric spaces of rank one via groups of Heisenberg type

1991 - 1998

Joint work with M. Cowling, A. Dooley, F. Ricci

The aim was too give an elementary unified approach to rank one symmetric spaces of the noncompact type. One starts the construction of these symmetric spaces by considering the H-type algebras satisfying the condition J_2.

One considers a nilpotent Lie algebra equiped with an inner product and an orthogonal decomposition

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}.$$

One introduces the mapping

$$J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$$

defined by

$$\langle J_Z X, X' \rangle = \langle Z, [X, X'] \rangle.$$

The Lie algebra $\mathfrak{n}$ is said to be of H-type if

$$J_Z^2 = -|Z|^2 I_d.$$

It satisfies the condition J_2 if for $X \in \mathfrak{v}$, $Z, Z' \in \mathfrak{z}$, there exists $Z'' \in \mathfrak{z}$ such that

$$J_Z J_{Z'} X = J_{Z''} X.$$

This leads to a simple description of the Riemannian symmetric spaces of the noncompact type which are of rank one.

Before ending I would like to say some personnal words.

During the conferences Adam Korányi is attending, by his presence, he is always driving exchanges between participants, talking with everybody.

But, in recent years, Adam Korányi is militantly in favour of the protection of the planet. As a result he does not fly anymore, and does not take part presently in conferences for which a long flight is necessary.

The writing of the book *Analysis on Symmetric Cones* took almost ten years. It was for me a beautiful time, thanks to the many exchanges we had regularly during that time.

Adam Korányi is a mathematician, a musician, *un homme des lumières*. Everybody who meets Adam Korányi appreciates his large culture, and his kindness.

Happy Birthday, Adam !