Korányi and the classification of homogeneous operators in the Cowen-Douglas class on $\mathbb{D}$

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Homogeneous operators

An operator $T : \mathcal{H} \to \mathcal{H}$ is homogeneous if

spectrum of $T \subset \overline{\mathbb{D}}$
$g(T)$ is unitarily equivalent to T for $g \in$ Möb = G_0

$$U_g^* T U_g = g(T) \; g \in G_0$$

T irreducible: $g \to U_g$ is a projective representation of G_0
$k \to U_k$ is a representation of K (rotation group)
$G = SU(1,1)$, $\widetilde{G}$, its universal cover

Multiplier representations

$\mathcal{H}$: RKHS of holomorphic functions on $\mathbb{D}$ with kernel $K(z, w)$

$$U_g f(z) = J(g^{-1}, z)\, f(g \cdot z) \quad f \in \mathcal{H},$$

unitary multiplier representation.

$$J(gh, z) = J(h, z)J(g, hz) \text{ (cocycle identity)}$$

Quasi invariance:

$$J(g, z)K(gz, gw)J(g, w)^* = K(z, w)\ g \in G_0$$

$Mf(z) = zf(z)$ is a homogeneous operator.

Examples

$\mathbb{A}^s(\mathbb{D})$ with kernel $K(z,w) = (1 - z\overline{w})^{-2s}$

$$\pi_s(g)F(z) = g'(z)^s \; F(gz) \; F \in \mathbb{A}^s(\mathbb{D})$$

and

$$Mf(z) = zf(z)$$

is homogeneous and irreducible.

New construction

Direct sum of two homogeneous operators is homogeneous, but not irreducible

$\Gamma_0 : \mathbb{A}^s(\mathbb{D}) \to \text{ Hol }(\mathbb{D}, \mathbb{C}^2)$, $\Gamma_1 : \mathbb{A}^{s+1}(\mathbb{D}) \to \text{ Hol }(\mathbb{D}, \mathbb{C}^2)$,

$$\Gamma_0 f_0 = \begin{pmatrix} f_0 \\ \frac{f_0'}{2s} \end{pmatrix} \qquad \Gamma_1 f_1 = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}$$

Transfer inner product

$$\langle \Gamma_0 f_0 + \Gamma_1 f_1, \Gamma_0 g_0 + \Gamma_1 g_1 \rangle = \langle f_0, g_0 \rangle + \frac{1}{\mu^2} \langle f_1, g_1 \rangle$$

We get a Hilbert space $H_{s,\mu}$.

New construction

Let $\Gamma = \Gamma_0 \oplus \mu\Gamma_1$, transfer the representations using Γ

$$\Gamma_0(\pi_s(g)f_0) = J(g,\cdot)(\Gamma_0 f_0) \circ g,$$

$$\Gamma_1(\pi_{s+1}(g)f_1) = J(g,\cdot)(\Gamma_1 f_1) \circ g$$

where $J(g,z)$ is the multiplier (2×2 matrix)

$$J(g,z) = \begin{pmatrix} g'^{s} & 0 \\ -cg'^{s+\frac{1}{2}} & g'^{s+1} \end{pmatrix}$$

Theorem

The operator $M^{s,\mu}$, multiplication by z is bounded, irreducible and homogeneous on $H_{s,\mu}$.

General construction

Rough idea

Towards a classification of homogeneous operators in $\mathbb{B}_k(\mathbb{D})$.

• Construct all $\widetilde{G}-$homogeneous holomorphic vector bundles via holomorphic induction

• Identify the ones with $\widetilde{G}$-invariant hermitian structure

• Determine the ones for which the hermitian structure comes from a reproducing kernel and obtain homogeneous operators that are in $B_k(\mathbb{D})$.

Holomorphic indcution

Basis for $\mathfrak{g}$:

$$X_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Basis for $\mathfrak{g}^{\mathbb{C}}$:

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$B = K^{\mathbb{C}}P^{-} = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C} \right\}$$

is a closed subgroup with Lie algbra $\mathfrak{b} = \mathbb{C}h + \mathbb{C}y$. Linear representations (ρ, V) of $\mathfrak{b}$ is a pair $[\rho(h), \rho(y)] = -\rho(y)$, is also a representation of K, hence give, by holomorphic induction all homogeneous holomorphic vector bundles

$\widetilde{G}$-invariant hermitian structure are given by $\rho(\widetilde{K})$-invariant inner product on V

$\rho(h)$ acts diagonally with real diagonal elements in a suitable basis (V will be identified with $\mathbb{C}^d$)

If V_λ is the eigenspace of $\rho(h)$ with eigenvalue ρ, $\rho(y)V_\lambda \subset V_{\lambda-1}$. Consequently $(\rho, \mathbb{C}^d)$ is determined by

$$\rho(h) = \begin{bmatrix} -\eta I_0 & & \\ & \ddots & \\ & & -(\eta + m) I_m \end{bmatrix}$$

where I_j is the identity on $\mathbb{C}^{d_j}$, $\mathbb{C}^d = \oplus_{j=0}^m \mathbb{C}^{d_j}$.

$$Y = \rho(y) = \begin{bmatrix} 0 & & & & \\ Y_1 & 0 & & & \\ & Y_2 & 0 & & \\ & & \ddots & & \\ & & & Y_m & 0 \end{bmatrix}$$

where $Y_j : \mathbb{C}^{d_{j-1}} \to \mathbb{C}^{d_j}$. Let $E^{(\eta, Y)}$ denote the holomorphic bundle induced by this representation. Let $[Y]$ be the equivalence class of Y under conjugation by block diagonal unitaries.

Theorem

The vector bundles $E^{(\eta,[Y])}$ *with* $\eta \in \mathbb{R}$, *form a parametrization of the elementary homogeneous holomorphic Hermitian vector bundles.* $E^{(\eta,[Y])}$ *is irreducible if and only if* Y *can not be split into orthogonal direct sum* $Y^{'} \oplus Y^{''}$ *with* $Y^{'}, Y^{''}$ *having the same block diagonal form as* Y.

Next, determine which ones above have the hermitian structure coming from a reproducing kernel. That is, which ones have a $\widetilde{G}$-invariant reproducing kernel $K(z, w)$. It suffices to enumerate the possibilities for $K(0, 0)$.

Recall, for $\lambda > 0$, $\mathbb{A}^{(\lambda)}(\mathbb{D})$ with the kernel $(1 - z\overline{w})^{-2\lambda}$. Let $\mathbb{C}^d = \oplus_{j=0}^m \mathbb{C}^{d_j}$. Write $F : \mathbb{D} \to \mathbb{C}^d$ as $F = (f_1, f_2, \cdots f_m)$ where $f_m : \mathbb{D} \to \mathbb{C}^{d_j}$. Let $\boldsymbol{A}^\eta = \oplus_{j=0}^m \mathbb{A}^{\eta+j}(\mathbb{D}) \otimes \mathbb{C}^{d_j}$. For $\eta > 0$, Y as before with $f_j \in \mathbb{A}^{(\eta+j)} \otimes \mathbb{C}^{d_j}$ define

$$(\Gamma^{(\eta,Y)} f_j)_\ell = \begin{cases} \frac{1}{(\ell-j)!} \frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} f_j^{(\ell-j)} & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j. \end{cases}$$

Let N be an invertible $d \times d$ block diagonal matrix with blocks N_j, $0 \leq j \leq m$, $d = d_0 + \cdots + d_m$, N_j positive definite, $N_0 = I_{d_0}$.
Write $\Gamma_N^{(\eta,Y)} = \Gamma^{(\eta,Y)} \circ N$, $\mathcal{H}_N^{(\eta,Y)}$ the image of $\Gamma_N^{(\eta,Y)}$.

Theorem

(A) The map $\Gamma_N^{(\eta,Y)}$ *is a* $\widetilde{G}$*-equivariant isomorphism of* $\boldsymbol{A}^\eta$ *onto the Hilbert space* $\mathcal{H}_N^{(\eta,Y)}$ *on which the* $\widetilde{G}$ *action is unitary via a multiplier (explicit). It has a reproducing kernel* $K_N^{(\eta,Y)}$ *(explicit).*
(B) The construction $\Gamma_N^{(\eta,Y)}$ *gives all elementary homogeneous holomorphic Hermitian vector bundles which possess a reproducing kernel, namely those of the form*

$$(E^{(\eta,Y)}, K_N^{(\eta,Y)}(0,0)^{-1})$$

where $\eta > 0$*,* Y *are arbitrary and* $K_N^{(\eta,Y)}(0,0)$ *is of the form given above.*

Theorem

All the homogeneous holomorphic Hermitian vector bundles with a reproducing kernel correspond to the homogeneous operators in the Cowen-Douglas class. The irreducible ones are the adjoint of the multiplication operator on the space $\mathcal{H}_l^{(\eta, Y)}$ for some $\eta > 0$ and irreducible Y.

References

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Thank you!