The Impact of Adam Korányi's Work on Jordan Theory

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July 2022

A real vector space A endowed with a commutative 'product' $(x, y) \mapsto x \circ y$ is called a **Jordan algebra** if the multiplication operators $L_x y := x \circ y$ satisfy the commutator identity

$$[L_x, L_{x^2}] = 0.$$

A Jordan algebra A is called **euclidean** if the symmetric bilinear form

$$(x|y) := \mathrm{tr}_A \ L_{x \circ y}$$

is positive definite. The basic examples are the self-adjoint $(r \times r)$-matrices $A = \mathcal{H}_r(\mathbf{K})$ over $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ (quaternions) endowed with the anti-commutator product

$$x \circ y = \frac{1}{2}(xy + yx).$$

If $r = 3$ one may also take the Cayley numbers $\mathbf{K} = \mathbf{O}$ to obtain the **exceptional Jordan algebra** $\mathcal{H}_3(\mathbf{O})$.

A complex vector space E endowed with a ternary composition

$$E \times E \times E \to E,\ (u,v,w) \mapsto \{uv^*w\}$$

which is symmetric bilinear in (u,w) and conjugate-linear in v is called a **Jordan triple** if the endomorphisms $L_{u,v}w := \{uv^*w\}$ satisfy the commutator identity

$$[L_{u,v}, L_{x,y}] = L_{\{uv^*x\},y} - L_{x,\{yu^*v\}}.$$

A Jordan triple E is called **hermitian** if the inner product

$$(u|v) := \mathrm{tr}_E\ L_{u,v}$$

is hermitian and positive definite. The basic examples are the matrix space $E = \mathbf{C}^{r\times s}$ endowed with the anti-commutator

$$\{uv^*w\} = \frac{1}{2}(uv^*w + wv^*u)$$

and its subtriples $\mathbf{C}^{r\times r}_{sym}$ and $\mathbf{C}^{s\times s}_{asym}$ of symmetric (resp. anti-symmetric) matrices.

If A is a euclidean Jordan algebra, then its complexification $E = A \oplus iA$ becomes a hermitian Jordan triple under

$$\{uv^*w\} := (u \circ v^*) \circ w + (w \circ v^*) \circ u - (u \circ w) \circ v^*$$

where $(x + iy)^* := x - iy$. Conversely, if E is a hermitian Jordan triple containing a 'unital' element $e \in E$, satisfying

$$L_{e,e} = \mathrm{id}_E,$$

then the self-adjoint part

$$A := \{x \in E : \ \{ex^*e\} = x\}$$

becomes a euclidean Jordan algebra under the product $x \circ y := \{xe^*y\}$. Such Jordan triples are said to be **of tube type**. For example, $E = \mathbf{C}^{r \times s}$ is of tube type if and only if $r = s$ (square matrices).

Siegel domain realization of bounded symmetric domains (Korányi-Wolf)

A **bounded symmetric domain** is a symmetric space $D = G/K$ of non-compact type, where G is a semi-simple real Lie group with maximal subgroup K having a non-discrete center. In the circular Harish-Chandra realization these domains arise exactly as the unit ball

$$D = \{z \in E : \ \|z\| < 1\}$$

of a hermitian Jordan triple E, with respect to the so-called spectral norm. Consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of the Lie algebra $\mathfrak{g}$ of G. Realizing $\mathfrak{g}$ as holomorphic vector fields $h(z)\frac{\partial}{\partial z}$ on D one shows that $\mathfrak{k}$ consists of linear vector fields, whereas

$$\mathfrak{p} = \{(v - \{zv^*z\})\frac{\partial}{\partial z} : \ v \in E\}.$$

The stabilizer group K at the origin $0 \in D$ consists of (linear) Jordan triple automorphisms of E. The **compact dual manifold** $M = U/K$ has the Lie algebra

$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$$

and

$$i\mathfrak{p} = \{(v + \{zv^*z\})\frac{\partial}{\partial z} : \ v \in E\}.$$

The matrix case $E = \mathbf{C}^{r\times s}$ yields the pseudo-unitary group $G = SU(r,s)$ acting by Moebius transformations

$$g(z) = (az+b)(cz+d)^{-1}.$$

Its compact dual is the Grassmannian $M = \mathrm{Gr}_r(\mathbf{C}^{r+s})$ with $U = SU(r+s)$.

An element $c \in E$ satisfying $\{cc^*c\} = c$ is called a **tripotent**. For matrices, these are the 'partial isometries'. Let S_ℓ denote the compact K-homogeneous manifold of all tripotents of rank $\ell \leq r$. According to Korányi-Wolf, the **Cayley transformation** induced by a tripotent c is

$$g_c := \exp\left(\frac{\pi}{4}(c + \{zc^*z\})\right)$$

This is a biholomorphic isometry of the compact dual space M. The original definition used the Harish-Chandra strongly orthogonal roots

$$\gamma_i(e_j - \{ze_j^*z\}) := \delta_{ij}$$

for $1 \leq i, j \leq r$, where $e_1, \ldots, e_r$ is a frame of minimal orthogonal tripotents. The range

$$g_c(D) = D_c$$

is the **Siegel domain** associated with c.

Any tripotent induces a **Peirce decomposition**

$$E = U \oplus V \oplus W$$

into eigenspaces of $L_{c,c}$ for eigenvalue $1, \frac{1}{2}, 0$ resp. In the matrix case $E = \mathbf{C}^{r\times s}$ and $c = \begin{pmatrix} 1_\ell & 0 \\ 0 & 0 \end{pmatrix}$ we have

$$E = \begin{pmatrix} \mathbf{C}^{\ell\times\ell} & \mathbf{C}^{\ell\times(s-\ell)} \\ \mathbf{C}^{(r-\ell)\times\ell} & \mathbf{C}^{(r-\ell)\times(s-\ell)} \end{pmatrix} = \begin{pmatrix} U & V \\ V & W \end{pmatrix}.$$

We say that c is **unital** if $E = U$ $(V = W = 0)$ and **maximal** if $E = U \oplus V$ $(W = 0)$. In general, the Peirce 2-space U is always of tube type. Therefore the self-adjoint part

$$A_c := \{x \in U : \ \{cx^*c\} = x\}$$

is a euclidean Jordan algebra with unit element c and product $x \circ y := \{xc^*y\}$. Let $\Omega_c \subset A_c$ denote the **symmetric cone** of A_c, i.e., the interior of the set of squares in A_c.

If c is **unital** then the Cayley transform is

$$g_c(u) = (c - u) \circ (c + u)^{-1}$$

and its range

$$D_c = \{u \in E : \ u + u^* \in \Omega_c\} = \Omega_c \oplus iA_c$$

is a **tube domain** (Siegel domain of first kind). This is a generalized (right) half-plane. If c is **maximal**, the Cayley transform is

$$g_c(u, v) = \Big((c - u) \circ (c + u)^{-1}, \sqrt{2}v \circ (c + u)^{-1}\Big)$$

and its range

$$D_c = \{u + v \in E = U \oplus V : \ \frac{u + u^*}{2} - \{cv^*v\} \in \Omega_c\}$$

is a Siegel domain (of second kind). If c is arbitrary, one obtains Siegel domains of third kind in a similar way. These domains are convex and unbounded (if $c \neq 0$.)

Poisson integral on symmetric domains

The tripotents of maximal rank r are precisely the **extreme points** of a bounded symmetric domain $D = G/K$ and form the **Shilov boundary** $S = S_r$. Every $g \in G$ has an analytic continuation onto a neighborhood of $\overline{D}$ which leaves S invariant. For any $z \in D$ the isotropy group

$$G^z := \{g \in G : \ g(z) = z\}$$

is compact and acts transitively on S. Let μ_z be the unique G^z-invariant probability measure on S. For $z = 0$ we have $G^0 = K$ and put $\mu := \mu_0$. The Radon-Nikodym derivative

$$\mathcal{P}(z,w) := \frac{\mu_z(dw)}{\mu(dw)}$$

is the **Poisson kernel** $\mathcal{P} : D \times S \to \mathbf{R}_+$. A similar construction holds in the unbounded setting of Siegel domains.

For $z, w \in E$ define the **Bergman endomorphism** on E by

$$B_{z,w}v = v - 2\{zw^*v\} + \{z\{wv^*w\}^*z\}.$$

Its determinant is a power

$$\det B_{z,w} = h(z,w)^p$$

of a sesqui-polynomial $h : E \times E \to \mathbf{C}$ of degree (r, r) called the **Jordan triple determinant**. Here p is the so-called 'genus'. For matrices $E = \mathbf{C}^{r \times s}$ we have

$$B_{z,w}v = (1_r - zw^*)v(1_s - w^*z)$$

and

$$h(z,w) = \det(1_r - zw^*) = \det(1_s - w^*z)$$

with $p = r + s$.

In the bounded setting, Korányi shows that the **Szegö kernel** (the reproducing kernel of the Hardy space $H^2(S)$) has the form

$$\mathcal{S}(z,w) = h(z,w)^{-d/r}$$

where $\frac{d}{r} = 1 + \frac{a}{2}(r-1) + b < p - 1 = 1 + a(r-1) + b$. After normalizing the Lebesgue measure on D the **Bergman kernel** is

$$\mathcal{K}(z,w) = h(z,w)^{-p}.$$

For the **Poisson kernel** he obtains

$$\mathcal{P}(z,w) = \frac{|\mathcal{S}(z,w)|^2}{\mathcal{S}(z,z)}$$

which shows the invariance under the action of G. Among other analytic properties, it is shown that for fixed $w \in S$ the function $\mathcal{P}_w(z) := \mathcal{P}(z,w)$ on D is **harmonic** in the sense of symmetric spaces, i.e., it is annihilated by all G-invariant differential operators (without constant term) on D.

Harmonic functions and Hua operator (Johnson-Korányi)

For a symmetric domain D (bounded or unbounded) the **Hua operator** $\mathcal{H}_D$ is a vector-valued operator on D which characterizes the Poisson integrals

$$f(z) = (\mathcal{P}\phi)(z) = \int_S \mu(d\zeta)\ \mathcal{P}(z,\zeta)\ \phi(\zeta)$$

of a (bounded) function ϕ on the Shilov boundary S by the **Hua equations** $\mathcal{H}_D f = 0$. More generally, the higher Poisson integrals

$$f(z) = (\mathcal{P}^s\phi)(z) = \int_S \mu(d\zeta)\ \mathcal{P}(z,\zeta)^s\ \phi(\zeta)$$

give rise to eigenfunctions

$$\mathcal{H}_D f = \left(\frac{d}{r}\right)^2\ \frac{s(s-1)}{4}\ f\ \mathrm{id}$$

Johnson-Korányi obtain Hua operators **with values in** $\mathfrak{k}^{\mathbf{C}} = L_{E,E}$ which satisfy the covariance condition

$$\mathcal{H}(f \circ g)(z) = g'(z)^{-1} \ (\mathcal{H}f)(gz) \ g'(z)$$

for all $g \in G$, with holomorphic derivative $g'(z) \in K^{\mathbf{C}}$. Let b_i be an orthonormal basis. Then

$$\mathcal{H}_T = \sum_{ij} L_{\{xb_ix\},b_j} \otimes \frac{\partial^2}{\partial z_i \partial \overline{z}_j}$$

on the tube domain $T = \Omega + iA, \ z = x + iy$, and for bounded domains

$$\mathcal{H}_D = \sum_{ij} L_{B(z,z)b_i,b_j} \otimes \frac{\partial^2}{\partial z_i \partial \overline{z}_j}.$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ consists of all left-invariant differential operators on G. Since $\mathfrak{g}$ contains the infinitesimal transvections $(u - \{zu^*z\})\frac{\partial}{\partial z}$ for all $u \in E$, the complex vector fields $u\frac{\partial}{\partial z}$ and $\{zv^*z\}\frac{\partial}{\partial z}$ for $u, v \in E$ belong to $\mathfrak{g}^{\mathbf{C}}$. Pulled back on G one obtains (for tube type)

$$\mathcal{H}_G^{(2)} = \sum_{i,j} L_{b_i,b_j} \otimes \left(b_j \frac{\partial}{\partial z} \cdot \{zb_i^* z\} \frac{\partial}{\partial z} \right) \in \mathfrak{k}^{\mathbf{C}} \otimes \mathcal{U}(\mathfrak{g}^{\mathbf{C}})$$

M. Lassalle found Hua operators **with values in** E. For Jordan triples one obtains a third order operator

$$\mathcal{H}_G^{(3)} = \sum_{i,j,k} \{b_i b_j^* b_k\} \otimes \left(\{z b_k^* z\} \frac{\partial}{\partial z} \cdot b_j \frac{\partial}{\partial z} \cdot \{z b_i^* z\} \frac{\partial}{\partial z} \right) \in E \otimes \mathcal{U}(\mathfrak{g}^{\mathbf{C}})$$

tube type domains, $e \in E$ unital tripotent

$$\mathcal{H}_G^{(2)} = \sum_{i,j} \{b_i b_j^* e\} \otimes \left(b_j \frac{\partial}{\partial z} \cdot \{z b_i^* z\} \frac{\partial}{\partial z} \right) \in E \otimes \mathcal{U}(\mathfrak{g}^{\mathbf{C}})$$

Lassalle's formula on D

$$\mathcal{H}_D^{(2)} = \sum_{i,j} \{e (B_{z,z}^{1/2} b_i)^* (B_{z,z}^{1/2} b_j)\} \otimes \frac{\partial^2}{\partial z_i \partial \overline{z}_j}$$

unbounded realization (tube domain) $z = x + iy, \ x \in \Omega$

$$\mathcal{H}_T^{(2)} = \sum_{i,j} \{x b_i^* b_j\} \otimes \frac{\partial^2}{\partial z_i \partial \overline{z}_j}$$

Holomorphic function spaces and reproducing kernels (Faraut-Korányi)

Under the natural action of K, the polynomial algebra $\mathcal{P}(E)$ has a multiplicity-free **Peter-Weyl decomposition** (Hua, Schmid, Kostant)

$$\mathcal{P}(E) = \sum_\lambda \mathcal{P}_\lambda(E)$$

into irreducible K-modules $\mathcal{P}_\lambda(E)$, where $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ is an arbitrary **integer partition** of length r. The finite-dimensional Hilbert space $\mathcal{P}_\lambda(E)$ has a **Fischer-Fock reproducing kernel** $\mathcal{E}_\lambda(z,w)$. For the unit ball of rank 1 we have $\mathcal{E}_m(z,w) = \frac{(z|w)^m}{m!}$. By definition, we have

$$e^{(z|w)} = \sum_\lambda \mathcal{E}_\lambda(z,w).$$

Let

$$(p|q) = \partial_p q(0) = \frac{1}{\pi^d} \int_E dz\ e^{-(z|z)}\ \overline{p(z)}\ q(z)$$

denote the **Fischer-Fock** inner product on $\mathcal{P}(E)$.

The **Faraut-Korányi binomial formula** is

$$h(z,w)^{-s} = \sum_{\lambda} (s)_\lambda \; \mathcal{E}_\lambda(z,w)$$

for all $s \in \mathbf{C}$, for the multi-variable **Pochhammer symbol**

$$(s)_\lambda := \prod_{j=1}^{r} (s - \frac{a}{2}(j-1))_{\lambda_j}.$$

Thus the weighted Bergman space $H_s^2(D)$ at parameter $s > p-1$ has inner product

$$(p|q)_s = \frac{1}{(s)_\lambda}(p|q)$$

for all $p, q \in \mathcal{P}_\lambda(E)$. Similarly, the Szego kernel has the expansion

$$\mathcal{S}(z,w) = h(z,w)^{-d/r} = \sum_{\lambda} (d/r)_\lambda \; \mathcal{E}_\lambda(z,w)$$

and the Hardy norm satisfies

$$(p|q)_S = \frac{1}{(d/r)_\lambda}(p|q)$$

As an application one obtains the **analytic continuation** of the scalar holomorphic discrete series (Wallach set) as a disjoint union of

$$s > \frac{a}{2}(r-1) \quad \text{(continuous Wallach set)},$$

$$s = \frac{a}{2}(\ell-1),\ \ell = 1,2,\ldots,r \quad \text{(discrete Wallach set)}.$$

$$N(e-x)^{-s} = \sum_{\lambda}(s)_\lambda \Phi_\lambda(x)$$

hypergeometric functions

$$\mathcal{F}_{p,q}\binom{s_1,\ldots,s_p}{t_1,\ldots,t_q}(z,w) = \sum_{\lambda} \frac{(s_1)_\lambda \cdots (s_p)_\lambda}{(t_1)_\lambda \cdots (t_q)_\lambda} \mathcal{E}_\lambda(z,w)$$

Korányi studied asymptotic behaviour, hypergeometric equations, Kummer relations etc.

Toeplitz operators on the Lie ball (Berger-Coburn-Korányi)

Let

$$h(z,w) = \sum_{i=0}^{r} h_i(z,w)$$

be the homogeneous expansion of the Jordan triple determinant and define sesqui-polynomials

$$F_j(z,w) = \sum_{i=0}^{j} \binom{r-i}{r-j} h_i(z,w).$$

Then

$$D = \{z \in E : \ F_j(z,z) > 0 \ \forall \ 1 \le j \le r\}$$

is defined by r analytic inequalities, with Shilov boundary

$$S = \{z \in E : \ F_j(z,z) = 0 \ \forall \ 1 \le j \le r\}.$$

Since $F_1(z,w) = r - (z|w)$ and $F_r(z,w) = h(z,w)$ we have $(z|z) < r$ and $h(z,z) > 0$ for all $z \in D$.

For the **spin factor** $E = \mathbf{C}^d$ of rank $r = 2$, we have $(z|w) = 2z \cdot \overline{w}$ and $N(z) = z \cdot z$. Hence

$$F_1(z,w) = 2 - (z|w) = 2(1 - z \cdot \overline{w}),$$

$$F_2(z,w) = 1 - (z|w) + N(z)\overline{N(w)} = 1 - 2z \cdot \overline{w} + z \cdot z \ \overline{w \cdot w}.$$

The associated **Lie ball**, is thus defined by two analytic inequalities

$$D = \{z \in \mathbf{C}^d : \ z \cdot \overline{z} < 1, \ 1 - 2z \cdot \overline{z} + |z \cdot z|^2 > 0\}.$$

This is an irreducible symmetric domain of rank $r = 2$. Its Shilov boundary

$$S = \{z \in \mathbf{C}^d : \ z \cdot \overline{z} = 1, \ 1 - 2z \cdot \overline{z} + |z \cdot z|^2 = 0\} = \mathbf{T} \cdot \mathbf{S}^{d-1}$$

is called the **Lie sphere**. On the other hand, the rank 1 tripotents are given by the co-sphere bundle

$$S_1 = \{\frac{x + i\xi}{2} : \ \|x\| = \|\xi\| = 1, \ x \cdot \xi = 0\} = \mathbf{S}^*(\mathbf{S}^{d-1})$$

Let $P : L^2(S) \to H^2(S)$ denote the Szegö projection onto the Hardy space over the Shilov boundary S. We define the **Toeplitz operator** T_f with (continuous) symbol $f \in \mathcal{C}(S)$ by

$$T_f(\phi) := P(f\phi)$$

for all $\phi \in H^2(S)$. Thus

$$T_f = PfP.$$

These are bounded operators on $H^2(S)$ which generate a (highly non-commutative) **Toeplitz C^*-algebra** $\mathcal{T}(S)$ containing the compact operators $\mathcal{K}(H^2(S))$.

If $r = 1$ (unit ball) then $\mathcal{T}/\mathcal{K} \approx \mathcal{C}(S)$ is commutative. For higher rank $r > 1$ the situation is much more complicated.

For every tripotent c of rank ℓ the **boundary component**

$$D_c := c + D \cap E_0(c) = c + \{w \in E_0(c) : \ \|w\| < 1\}$$

corresponds to a bounded symmetric domain of rank $r - \ell$, with Shilov boundary

$$S_c := c + S_{r-\ell} \cap E_0(c).$$

Let T^c denote the 'little' Toeplitz operator acting on the 'little' Hardy space $H^2(S_c)$.

Theorem

For every tripotent c the Toeplitz C^*-algebra $\mathcal{T}(S)$ has an irreducible representation

$$\sigma_c : \mathcal{T}(S) \to \mathcal{T}(S_c) \subset \mathcal{L}(H^2(S_c))$$

which is uniquely determined by the property

$$\sigma_c(T_f) = T^c_{f_c}$$

for all $f \in \mathcal{C}(S)$, where $f_c(\zeta) := f(c + \zeta)$. These representations are pairwise inequivalent.

With more effort it is proved that the representations σ_c, for tripotents c of rank $0 \le \ell \le r$, constitute the full spectrum of $\mathcal{T}(S)$

$$\mathrm{Spec}\mathcal{T}(S) = \bigcup_{0 \le \ell \le r} S_\ell$$

endowed with a stratified non-Hausdorff topology. This results in a **composition sequence** of C^*-ideals

$$\mathcal{K} = \mathcal{I}_1 \subset \mathcal{I}_2 \subset \ldots \subset \mathcal{I}_r \subset \mathcal{T}(S)$$

of length r such that $\mathcal{I}_{\ell+1}/\mathcal{I}_\ell$ is stably isomorphic to $\mathcal{C}(S_\ell)$. In particular, for the Lie sphere $(r = 2)$ one obtains

$$\mathcal{K} \subset \mathcal{I}_2 \subset \mathcal{T}(S)$$

and the ideal

$$\mathcal{I}_2 = \mathcal{K}(H^2(\mathbf{T})) \otimes \mathcal{C}\mathcal{Z}(\mathbf{S}^{d-1})$$

is given by **Calderon-Zygmund operators** (pseudo-differential operators of order 0) on $\mathbf{S}^{d-1}$