The monopole-dimer model for Cartesian products of plane graphs

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Loop-vertex Model

The loop-vertex model on the (edge- and vertex-weighted) graph G with an orientation $\mathcal{O}$ is the collection $\mathcal{L}$ of configurations consisting directed even loops, doubled edges and some isolated vertices with the weight of each configuration defined as:

$$w(C) = \prod_{\ell = \text{loop in } C} w(\ell) \prod_{\substack{v \text{ an} \\ \text{isolated vertex} \\ \text{in } C}} x(v),$$

$$\text{where } w(\ell) = -\prod_{i=1}^{2m} sgn(v_i, v_{i+1})\, a_{v_i, v_{i+1}} \quad \text{for } \ell = (v_1, v_2, .., v_{2m}, v_1).$$

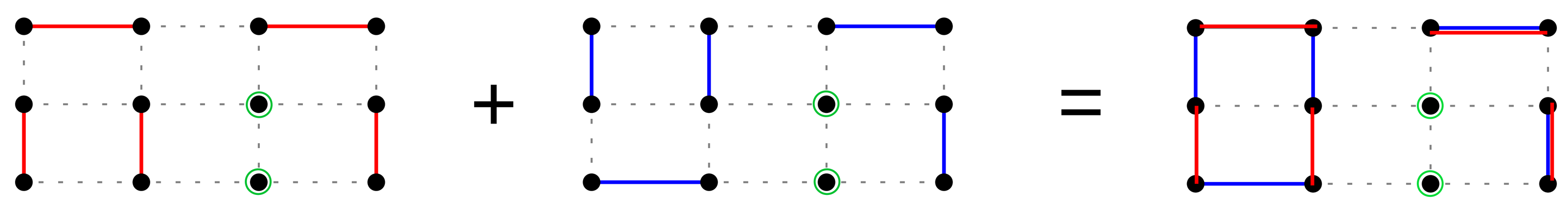

Figure: Two matchings overlapping to form a loop-vertex configuration.

Partition function

Partition function of the loop-vertex Model on $(G, \mathcal{O})$ is defined as

$$\mathcal{Z}_{G,\mathcal{O}} := \sum_{C \in \mathcal{L}} w(C).$$

Monopole-Dimer model

The weight of the directed loop $\ell = (v_1, v_2, .., v_{2m}, v_1)$ in a plane graph G with a Pfaffian orientation can be expressed as follows

$$w(\ell) = (-1)^{\#\text{ of vertices enclosed by } \ell} \prod_{j=1}^{2m} a_{v_j, v_{j+1}}.$$

Partition function is a determinant

The partition function [2] of the monopole-dimer model is the determinant of the generalised adjacency matrix, $\mathcal{K}_G$ of $(G, \mathcal{O})$ defined as

$$\mathcal{K}_G(v, v') = \begin{cases} x(v) & v = v', \\ a_{v,v'} & v \to v' \text{ in } \mathcal{O}, \\ -a_{v,v'} & v' \to v \text{ in } \mathcal{O}, \\ 0 & (v, v') \notin E(G). \end{cases} \tag{1}$$

Consequetly, $\det \mathcal{K}_G$ becomes independent of the Pfaffian orientation.

Sign of a cycle decomposition

A *cycle decomposition* of an even graph G is a family $\mathcal{D}$ consisting of cycles in G such that each edge in G is covered in exactly one cycle. The *sign* of a cycle decomposition $\mathcal{D} = \{c_1, c_2, \ldots, c_k\}$ of an even plane graph G is given by

$$\text{sgn}(\mathcal{D}) := \prod_{i=1}^{k} (-1)^{1 + \#\text{ of vertices in } V(G) \text{ enclosed by } c_i}. \tag{2}$$

Key Tool

Let G be a connected, bipartite, even plane graph. Then all cycle decompositions of G have the same sign.

(Extended) monopole-dimer model

The *i-projection* of a subgraph S of $G_1 \square G_2 \square \cdots \square G_k$, denoted as $\tilde{S}_i$, is the graph obtained by contracting all but G_i-edges of S.

Let $G_1, \ldots, G_k$ be k simple plane naturally labelled bipartite graphs with Pfaffian orientations and L be their oriented Cartesian product. Let $\ell = (w_0, w_1, \ldots, w_{2k-1}, w_{2m} = w_0)$ be a directed even loop in L, and $\mathcal{L}_i$ be a cycle decomposition of the i-projection $\tilde{\ell}_i$. For $i \in [k]$, let $G^{(i)}$ be the graph $G_1 \square \cdots \square G_{i-1} \square G_{i+1} \square \cdots \square G_k$. For $\hat{v} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \in V(G^{(i)})$, let $G_i(\hat{v})$ be the copy of G_i in L corresponding to $\hat{v}$ and let $e_i(\hat{v})$ be the number of edges lying both in ℓ and $G_i(\hat{v})$. Now let

$$e_i = \sum_{\substack{\hat{v} \in V(\hat{G}^{(i)}) \\ v_{i+1} + \cdots + v_k + (k-i) \equiv 1 \pmod 2}} e_i(\hat{v}).$$

Then the *sign* of ℓ is defined by

$$\text{sgn}(\ell) := -\prod_{i=1}^{k-1} (-1)^{e_i} \prod_{i=1}^{k} \text{sgn}(\mathcal{L}_i). \tag{3}$$

Now suppose that L has been given vertex weights $x(w)$ for $w \in V(L)$ and edge weights a_e for $e \in E(L)$. Then the *weight of the loop* ℓ is expressed as

$$w(\ell) = \text{sgn}(\ell) \prod_{i=1}^{2m} a_{w_i, w_{i+1}}. \tag{4}$$

Theorem

Let $G_1, G_2, \ldots, G_k$ be k simple plane bipartite graphs with Pfaffian orientations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ respectively. The partition function [3] of the monopole-dimer model for the weighted oriented Cartesian product L of $G_1, G_2, \ldots, G_k$ is given by

$$\mathcal{Z}_L = \det \mathcal{K}_G, \tag{5}$$

where $\mathcal{K}_G$ is the generalised adjacency matrix defined in (1) for L.

Grid graphs

Let $G = P_{2m_1} \square \cdots \square, P_{2m_d}$ be a d-dimensional grid graph with the boustrophedon labelling. Let vertex weights be x for all vertices of G, edge weights be a_i for the P_{2m_i}-edges. Then the partition function of the monopole-dimer model on G is given by

$$\prod_{i_d=1}^{m_d} \cdots \prod_{i_1=1}^{m_1} \left(x^2 + \sum_{q=1}^{d} 4a_s^2 \cos^2 \frac{i_q \pi}{2m_q + 1} \right)^{2^{d-1}}.$$

In particular, partition function of the monopole-dimer model on the three-dimensional grid with even grid lengths is

$$\prod_{j=1}^{m_3} \prod_{s=1}^{m_2} \prod_{k=1}^{m_1} \left(x^2 + 4a_1^2 \cos^2 \frac{\pi k}{2m_1+1} + 4a_2^2 \cos^2 \frac{\pi s}{2m_2+1} + 4a_3^2 \cos^2 \frac{\pi j}{2m_3+1} \right)^4.$$

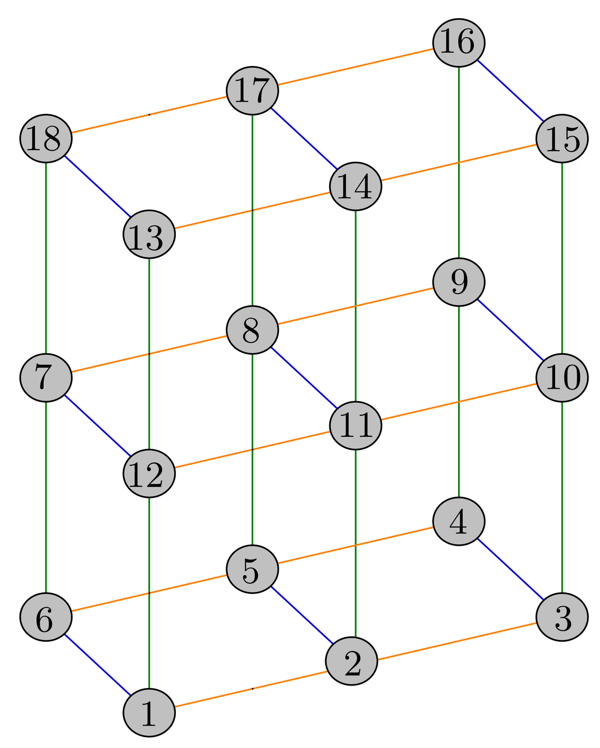

Figure: The boustrophedon labelling on $P_3 \square P_2 \square P_3$.

Observations

- $\det \mathcal{K}_G$ in (5) becomes independent of the Pfaffian orientations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$.
- Partition function for d-dimensional grids is always a polynomial in x and a_i's with nonnegative integer coefficients.
- Moreover, the partition function is the $2^{(d-1)}$-th power when grid lengths are even.

Future scope

- If the partition function of a graph G is p^4 or $p^{2^{(d-1)}}$ for some polynomial p, then does p carries some information about the graph G?
- The extension for the non-bipartite case has also been achieved now. Further extension to non-planar graphs would be interesting.

References

[1] P. W. Kasteleyn. Dimer statistics and phase transitions. *Journal of Mathematical Physics*, 4(2):287–293, 1963.

[2] Arvind Ayyer. A statistical model of current loops and magnetic monopoles. *Mathematical Physics, Analysis and Geometry*, 18(1), Jun 2015.

[3] Anita Arora and Arvind Ayyer. The monopole-dimer model on Cartesian products of plane graphs. *arXiv preprint arXiv:2205.11791*, 2022.