Affine Semigroups of Maximal Projective Dimension

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Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and non-negative integers respectively. Finitely generated submonoids of $\mathbb{N}^d$ are known as affine semigroups. If $d = 1$, affine semigroups correspond to numerical semigroups. A submonoid S of $\mathbb{N}$ is called a numerical semigroup if $\mathbb{N} \setminus S$ is finite. If $S \neq \mathbb{N}$ then the finiteness of $\mathbb{N} \setminus S$ implies that there exist at least one element f in $\mathbb{N} \setminus S$ such that $f + S \setminus \{0\} \subset S$. Such elements are called pseudo-Frobenius elements and the maximum of these elements is called the Frobenius number of S. Clearly, this does not hold for affine semigroups in general because an affine semigroup may not be finitely complemented in $\mathbb{N}^d$. In this article, we consider such affine semigroups where the set of pseudo-Frobenius elements is non-empty. Such semigroups are called MPD-semigroups. We generalize the notion of symmetric numerical semigroups, pseudo-symmetric numerical semigroups to the case of MPD-semigroups in $\mathbb{N}^d$. We prove that under suitable conditions these semigroups satisfy the Extended Wilf's conjecture.

Preliminaries

Let S be a finitely generated submonoid of $\mathbb{N}^d$, say generated by $\{a_1, a_2, \ldots, a_n\} \subset \mathbb{N}^d$. Such semigroups are called affine semigroups. Consider the cone of S in $\mathbb{Q}^d_{\geq 0}$,

$$\operatorname{cone}(S) = \left\{ \sum_{j=1}^n \lambda_j a_j \mid \lambda_j \in \mathbb{Q}_{\geq 0},\ j \in [1, n] \right\}$$

and define $\mathcal{H}(S) = (\operatorname{cone}(S) \setminus S) \cap \mathbb{N}^d$.

- An element $f \in \mathcal{H}(S)$ such that $f + s \in S$ for all $s \in S \setminus \{0\}$, is called a pseudo-Frobenius element of S. The set of pseudo-Frobenius elements of S,

  $$\operatorname{PF}(S) = \{ f \in \mathcal{H}(S) \mid f + n_j \in S, \forall j \in [1, n] \}.$$

  Note that for a finitely generated submonoid S, $\operatorname{PF}(S)$ may be empty also.
- If $\mathcal{H}(S)$ is finite and non-empty then S is called a $\mathcal{C}$-semigroup. In $\mathcal{C}$-semigroups the existence of pseudo-Frobenius elements is guaranteed, i.e $\operatorname{PF}(S) \neq \emptyset$.

On $\mathcal{H}(S)$, we define a relation : $\mathbf{x} \leq \mathbf{y}$ iff $\mathbf{y} - \mathbf{x} \in S$. It is a partial order (reflexive, transitive and anti-symmetric) on $\mathcal{H}(S)$.

Theorem: Let S be an affine semigroup in $\mathbb{N}^d$ such that $\mathcal{H}(S)$ is finite. Then

(i) $\operatorname{PF}(S) = \operatorname{Maximals}_{\leq} \mathcal{H}(S)$.

(ii) Let $\mathbf{x} \in \mathbb{N}^d$. Then $\mathbf{x} \in \mathcal{H}(S)$ if and only if $f - \mathbf{x} \in S$ for some $f \in \operatorname{PF}(S)$.

Affine semigroups of maximal projective dimension

Let k be a field. The semigroup ring $k[S]$ of S is a k-subalgebra of the polynomial ring $k[t_1, \ldots, t_d]$. In other words, $k[S] = k[\mathbf{t}^{a_1}, \ldots, \mathbf{t}^{a_n}]$, where $\mathbf{t}^{a_i} = t_1^{a_{i1}} \cdots t_d^{a_{id}}$ for $a_i = (a_{i1}, \ldots, a_{id})$ and for all $i = 1, \ldots, n$. Set $R = k[x_1, \ldots, x_n]$ and define a map $\pi : R \to k[S]$ given by $\pi(x_i) = \mathbf{t}^{a_i}$ for all $i = 1, \ldots, n$. Set $\deg x_i = a_i$ for all $i = 1, \ldots, n$. Observe that R is a multi-graded ring and that π is a degree preserving surjective k-algebra homomorphism. We denote by I_S the kernel of π. Then I_S is a homogeneous ideal, generated by binomials. A binomial $\phi = \prod_{i=1}^n x_i^{\alpha_i} - \prod_{j=1}^n x_j^{\beta_j} \in I_S$ if and only if $\sum_{i=1}^n \alpha_i a_i = \sum_{j=1}^n \beta_j a_j$.

- An affine semigroup S satisfies the **maximal projective dimension** (MPD) if $\operatorname{pdim}_R k[S] = n - 1$.
- (J. I Garcia-Garcia et. al., 2020) S is an MPD-semigroup if and only if $\operatorname{PF}(S) \neq \emptyset$.
- If S is an MPD-semigroup then $b \in S$ is the S-degree of the $(n-2)$th minimal syzygy of $k[S]$ if and only if $b \in \{a + \sum_{i=1}^n a_i \mid a \in \operatorname{PF}(S)\}$.

Let $S = \langle n_1 = (2, 11), n_2 = (3, 0), n_3 = (5, 9), n_4 = (7, 4) \rangle$. Then we have a minimal free resolution of $k[S]$,

$$0 \to R(-(81, 93)) \oplus R(-(94, 82)) \to R^6 \to R^5 \to R \to k[S] \to 0.$$

Therefore, $\operatorname{pdim}_R k[S] = 3$. Hence, S is MPD and we have

$$\operatorname{PF}(S) = \{(81, 93) - \sum_{i=1}^4 n_i, (94, 82) - \sum_{i=1}^4 n_i\}.$$

Therefore, $\operatorname{PF}(S) = \{(64, 89), (77, 58)\}$.

$\prec$-Symmetric and $\prec$-Pseudo-symmetric Semigroups

A term order (also known as monomial ordering) on $\mathbb{N}^d$ is a total order compatible with the addition of $\mathbb{N}^d$. Let $\prec$ be a term order on $\mathbb{N}^d$ then $\operatorname{F}(S)_\prec = \max_\prec \mathcal{H}(S)$, if it exists, is called a Frobenius element of S. In particular,

$$\operatorname{F}(S) = \{ \operatorname{F}(S)_\prec = \max_\prec \mathcal{H}(S) \mid \prec \text{ is a term order } \}.$$

- Frobenius elements of S may not exist. However, if $\mathcal{H}(S)$ is finite, then S has Frobenius element.
- Every Frobenius element of S is a pseudo-Frobenius element, i.e., $\operatorname{F}(S) \subseteq \operatorname{PF}(S)$.
- In the case $\operatorname{F}(S) \neq \emptyset$, we fix a term order $\prec$ such that $\operatorname{F}(S)_\prec = \max_\prec \mathcal{H}(S) \in \operatorname{F}(S)$. If $\operatorname{PF}(S) = \{\operatorname{F}(S)_\prec\}$, then S is called a **$\prec$-symmetric semigroup** and if $\operatorname{PF}(S) = \{\operatorname{F}(S)_\prec, \operatorname{F}(S)_\prec/2\}$, then S is called **$\prec$-pseudo-symmetric semigroup**.

Characterizations of $\prec$-Symmetric and $\prec$-Pseudo-symmetric Semigroups

Theorem: Let S be a $\mathcal{C}$-semigroup and $\operatorname{F}(S)_\prec$ denote the Frobenius element of S with respect to an order $\prec$. Then

(i) S is a $\prec$-symmetric semigroup if and only if for each $g \in \operatorname{cone}(S) \cap \mathbb{N}^d$ we have:

$$g \in S \iff \operatorname{F}(S)_\prec - g \notin S.$$

(ii) S is a $\prec$-pseudo-symmetric semigroup if and only if $\operatorname{F}(S)_\prec$ is even and for each $g \in \operatorname{cone}(S) \cap \mathbb{N}^d$ we have:

$$g \in S \iff \operatorname{F}(S)_\prec - g \notin S \quad \text{and} \quad g \neq \operatorname{F}(S)_\prec/2.$$

On $\operatorname{cone}(S)$, we define a relation $\leq_c$ as follows: $g \leq_c f$ if $g_i \leq f_i$ for all $i \in [1, d]$.

Theorem: Let S be a $\mathcal{C}$-semigroup such that $\operatorname{cone}(S) \cap \mathbb{N}^d = \mathbb{N}^d$. Then

(i) S is $\prec$-symmetric if and only if $|\mathcal{H}(S)| = |\{g \in S \mid g \leq_c \operatorname{F}(S)_\prec\}|$.

(ii) S is $\prec$-pseudo-symmetric if and only if $|\mathcal{H}(S) \setminus \{\operatorname{F}(S)_\prec/2\}| = |\{g \in S \mid g \leq_c \operatorname{F}(S)_\prec\}|$ and $\operatorname{F}(S)_\prec$ is even.

Extended Wilf's Conjecture

Let S be a $\mathcal{C}$-semigroup. Define the Frobenius number of S as $\mathcal{N}(\operatorname{F}(S)_\prec) = |\mathcal{H}(S)| + |\{g \in S \mid g \prec \operatorname{F}(S)_\prec\}|$.

- Let S be a $\mathcal{C}$-semigroup. The **extended Wilf's conjecture** is

  $$|\{g \in S \mid g \prec \operatorname{F}(S)_\prec\}| \cdot e(S) \geq \mathcal{N}(\operatorname{F}(S)_\prec) + 1,$$

where $e(S)$ denotes the embedding dimension of S.

Theorem: Let S be a $\mathcal{C}$-semigroup such that $\operatorname{cone}(S) \cap \mathbb{N}^d = \mathbb{N}^d$. Then for $\prec$-symmetric and $\prec$-pseudo-symmetric semigroups, the extended Wilf's conjecture holds.

RF-matrices and Generic toric ideals

Let $S = \langle a_1, \ldots, a_n \rangle$ be a MPD-semigroup in $\mathbb{N}^d$, minimally generated by $a_1, \ldots, a_n$. Let $f \in \operatorname{PF}(S)$. An $n \times n$ matrix $M = (m_{ij})$ is a **Row-Factorization matrix** (RF-matrix) of f if $m_{ii} = -1$ for every i, $m_{ij} \in \mathbb{N}$ if $i \neq j$ and for every $i = 1, \ldots, n$, $\sum_{j=1}^n m_{ij} a_j = f$.

Let $I_S \subset k[x_1, \ldots, x_n]$ be the defining ideal of the semigroup ring $k[S]$. Then $I_S \subset k[x_1, \ldots, x_n]$ is called **generic** if it is minimally generated by the binomials of full support.

Theorem: Let S be a MPD-semigroup. If I_S is generic, then $\operatorname{RF}(f) = (m_{ij})$ is unique for each $f \in \operatorname{PF}(S)$ and $m_{ij} \neq m_{i'j}$ for all $i \neq i'$.

Gluing of MPD-semigroups

Let $S \subseteq \mathbb{N}^d$ be an affine semigroup and $G(S)$ be the group spanned by S, that is, $G(S) = \{a - b \in \mathbb{Z}^d \mid a, b \in S\}$. Let A be the minimal generating system of S and $A = A_1 \amalg A_2$ be a nontrivial partition of A . Let S_i be the submonoid of $\mathbb{N}^d$ generated by $A_i, i \in 1, 2$. Then $S = S_1 + S_2$. We say that S is the **gluing** of S_1 and S_2 by d if $d \in S_1 \cap S_2$ and $G(S_1) \cap G(S_2) = d\mathbb{Z}$.

Theorem: Let S be a gluing of MPD-semigroups S_1 and S_2 by d. Then

$$\operatorname{PF}(S) = \{ f + g + d \mid f \in \operatorname{PF}(S_1),\ g \in \operatorname{PF}(S_2)\}.$$

Theorem: Let $n \geq 3$ and $S = \langle a_1, \ldots, a_n \rangle$ be a gluing of MPD-semigroups. Then I_S is not generic.

References

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