Troupes, Cumulants, and Stack-Sorting

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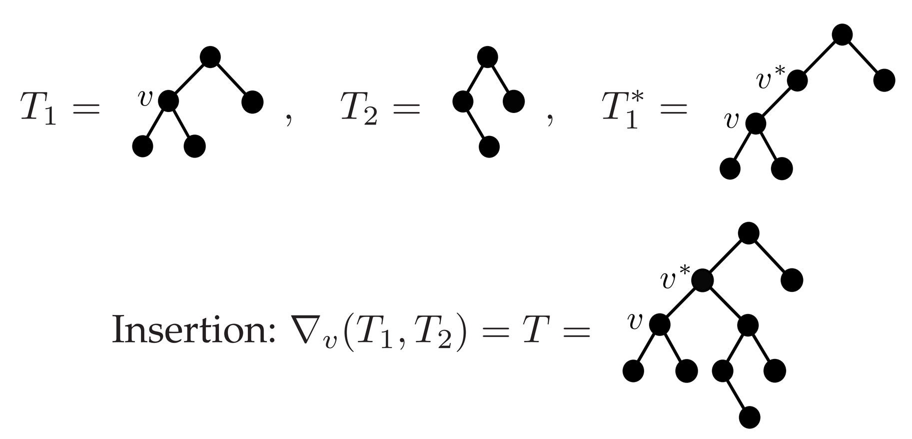
Abstract

In several cases, a sequence of free cumulants that counts certain binary plane trees corresponds to a sequence of classical cumulants that counts the decreasing versions of the same trees. Using two new operations on binary plane trees that we call *insertion* and *decomposition*, we show that this surprising phenomenon holds for families of trees that we call *troupes*. The proof relies on two new formulas, each of which is given as a sum over objects called *valid hook configurations*. The first of these formulas provides detailed information about the preimages of a permutation under the postorder traversal whose underlying trees belong to a given troupe; the second is a new combinatorial formula that converts from a sequence of free cumulants to the corresponding sequence of classical cumulants. The unexpected connection between troupes and cumulants provides a powerful new tool for analyzing the stack-sorting map s (which is defined via the postorder traversal) that hinges on free probability theory. We give numerous applications of this method. For example, we show that if $\sigma \in S_{n-1}$ is chosen uniformly at random and des denotes the descent statistic, then the expected value of $\mathrm{des}(s(\sigma))+1$ is

$$\left(3-\sum_{j=0}^{n}\frac{1}{j!}\right)n.$$

Furthermore, the variance of $\mathrm{des}(s(\sigma))+1$ is asymptotically $(2+2e-e^2)n$. We obtain similar results concerning the expected number of descents of postorder traversals of decreasing binary plane trees of various types. We also obtain improved estimates for $|s(S_n)|$ and an improved lower bound for the degree of noninvertibility of $s : S_n \to S_n$.

Troupes



Decomposition: $\Delta_{v^*}(T) = (T_1, T_2)$

A collection $\mathbf{T}$ of binary plane trees is a *troupe* if it is closed under insertion and decomposition. Let $\mathbf{T}_n$ be the set of trees in $\mathbf{T}$ with n vertices.

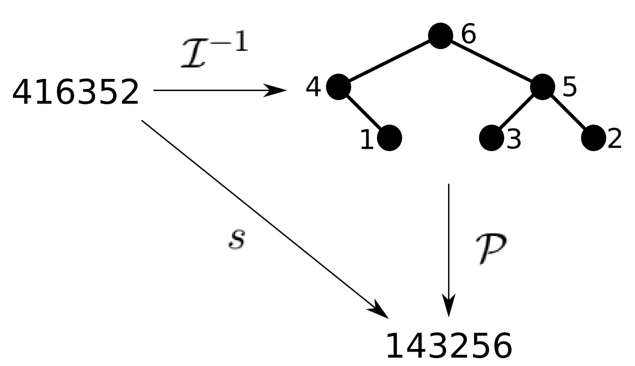
The set of all binary plane trees is a troupe.

A binary plane tree is *full* if no vertex has exactly 1 child. The set of full binary plane trees is a troupe.

A *Motzkin tree* is a binary plane tree such that every vertex with exactly 1 child actually has a left child. The set of Motzkin trees is a troupe.

Tree Traversals and Stack-Sorting

A *decreasing binary plane tree* is a binary plane tree whose vertices are bijectively labeled with the elements of $[n]$ so that the labels decrease along every path. Given a set $\mathbf{T}$ of binary plane trees, let $\mathrm{D}\mathbf{T}$ be the set of decreasing binary plane trees such that deleting the labels yields a tree in $\mathbf{T}$. The *in-order traversal* $\mathcal{I}$ and *postorder traversal* $\mathcal{P}$ are two maps from decreasing binary plane trees to permutations. The map $\mathcal{I}$ is bijective, and the *stack-sorting map* is the map $s = \mathcal{P} \circ \mathcal{I}^{-1}$.



Cumulants

Let R be a commutative ring. Given a sequence $(u_n)_{n\geq 1}$ of elements of R and a set partition ρ, let

$$(u_\bullet)_\rho = \prod_{B\in\rho} u_{|B|}.$$

Consider a sequence $(m_n)_{n\geq 1}$ of elements of R, called a *moment sequence*. There is an associated sequence $(c_n)_{n\geq 1}$ of *classical cumulants* defined implicitly by

$$m_n = \sum_{\substack{\rho \text{ a partition}\\ \text{of } \{1,\ldots,n\}}} (c_\bullet)_\rho.$$

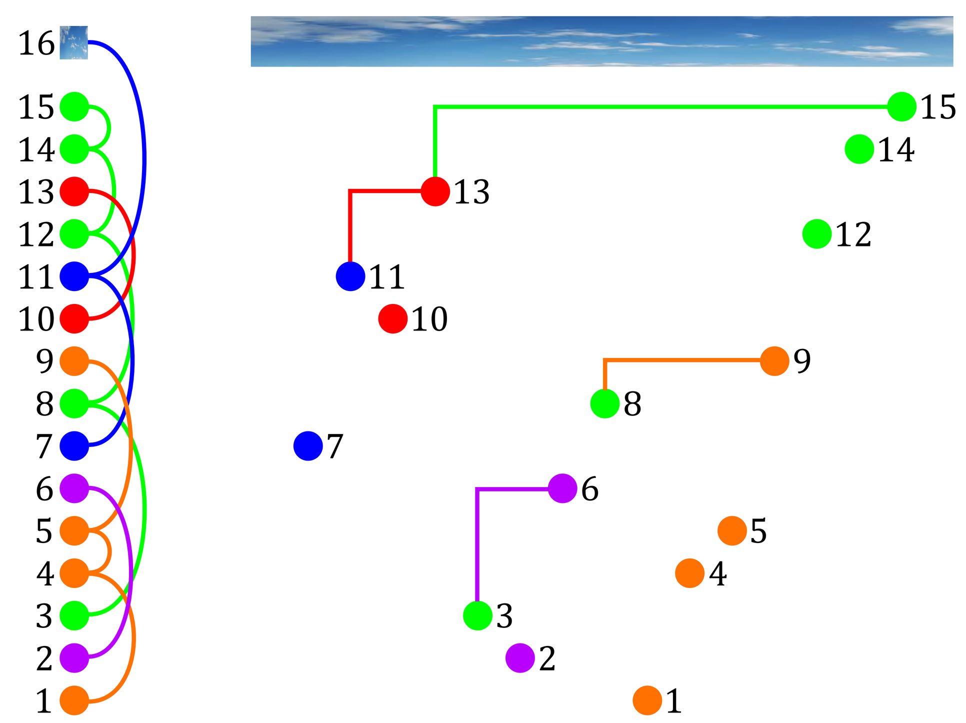
There is also an associated sequence of *free cumulants* $(\kappa_n)_{n\geq 1}$ defined implicitly by

$$m_n = \sum_{\substack{\rho \text{ a noncrossing}\\ \text{partition of } \{1,\ldots,n\}}} (\kappa_\bullet)_\rho.$$

Valid Hook Configurations

Start by plotting a permutation π. Draw "hooks" on the plot so that

1. the descent tops of the plot of π are precisely the southwest endpoints of the hooks;
2. no hook passes underneath a point in the plot;
3. hooks do not cross or overlap.



This produces a *valid hook configuration* $\mathcal{H}$, which induces a coloring, which induces a set partition $|\mathcal{H}$. Given a set S of permutations, let $\mathrm{VHC}(S)$ be the set of valid hook configurations of permutations in S.

Main Formulas

Theorem ([1]). *If $\mathbf{T}$ is a troupe and $\pi \in S_n$, then*

$$|\mathcal{P}^{-1}(\pi)\cap \mathrm{D}\mathbf{T}| = \sum_{\mathcal{H}\in\mathrm{VHC}(\pi)} (|\mathbf{T}_{\bullet-1}|)_{|\mathcal{H}}.$$

Theorem ([1]). *If $(\kappa_n)_{n\geq1}$ and $(c_n)_{n\geq1}$ are associated sequences of free and classical cumulants, respectively, then*

$$-c_n = \sum_{\mathcal{H}\in\mathrm{VHC}(S_{n-1})} (-\kappa_\bullet)_{|\mathcal{H}}.$$

Applications

Theorem ([1]). *Let $\mathbf{T}$ be a troupe. If we define a sequence of free cumulants by $\kappa_n = -|\mathbf{T}_{n-1}|$, then the corresponding classical cumulants are given by $c_n = -|\mathrm{D}\mathbf{T}_{n-1}|$.*

Theorem ([1]). *If $\mathcal{T}$ is a random decreasing binary plane tree with $n-1$ vertices, then*

$$\mathbb{E}[\mathrm{des}(\mathcal{P}(\mathcal{T}))+1] = \left(3-\sum_{j=0}^{n}\frac{1}{j!}\right)n \sim (3-e)n.$$

Moreover, $\mathrm{Var}[\mathrm{des}(\mathcal{P}(\mathcal{T}))+1] \sim (2+2e-e^2)n$.

Theorem ([1]). *If $\mathcal{T}$ is a random decreasing full binary plane tree with $n-1$ vertices, then*

$$\mathbb{E}[\mathrm{des}(\mathcal{P}(\mathcal{T}))+1] \sim \left(1-\frac{2}{\pi}\right)n.$$

Theorem ([1]). *If $\mathcal{T}$ is a random decreasing Motzkin tree with $n-1$ vertices, then*

$$\mathbb{E}[\mathrm{des}(\mathcal{P}(\mathcal{T}))+1] \sim \left(1-\frac{3\sqrt{3}}{2\pi}\left(e^{\frac{\pi}{3\sqrt{3}}}-1\right)\right)n.$$

Say a permutation is *uniquely sorted* if it has exactly 1 preimage under the stack-sorting map.

Theorem ([2]). *There are no uniquely sorted permutations of even size. Uniquely sorted permutations of odd size are counted by Lassalle's sequence, which is the sequence of absolute values of the classical cumulants of the standard semicircular distribution.*

Theorem ([1]). *Let $\mathrm{hook}(\mathcal{H})$ denote the number of hooks in a valid hook configuration $\mathcal{H}$. Then*

$$\sum_{n\geq1}\sum_{\mathcal{H}\in\mathrm{VHC}(S_{n-1})} x^{\mathrm{hook}(\mathcal{H})+1}\frac{z^n}{n!} = -\log\left(1-x\int\frac{e^{(1-x)z}J_1(2z\sqrt{x})}{z\sqrt{x}}dz\right),$$

where J_1 is a Bessel function of the first kind and the integral is taken so that it approaches 0 as $z\to\infty$.

Theorem ([1]). *The limit* $\lim_{n\to\infty}\left(\frac{|s(S_n)|}{n!}\right)^{1/n}$ *exists and lies in the interval* $[0.68631, 0.75260]$.

The *degree of noninvertibility* (see [3]) of a function $f\colon X\to X$ is defined by

$$\deg(f\colon X\to X) = \frac{1}{|X|}\sum_{x\in X}|f^{-1}(x)|^2.$$

Theorem ([1, 3]). *The limit* $\lim_{n\to\infty}(\deg(s\colon S_n\to S_n))^{1/n}$ *exists and lies in the interval* $[1.62924, 4]$.

Suggestions for Future Work

Question ([1]). *Let $\mathcal{T}$ be a random decreasing binary plane tree with n vertices. What can be said about the expected number of peaks of $\mathcal{P}(\mathcal{T})$? What about other statistics?*

Conjecture ([1]). *The polynomial* $\sum_{\sigma\in S_{n-1}} x^{\mathrm{des}(s(\sigma))+1}$ *has only real roots; consequently, its coefficients are unimodal.*

References

[1] C. Defant, Troupes, cumulants, and stack-sorting. *Adv. Math.*, **399** (2022).

[2] C. Defant, M. Engen, and J. A. Miller, Stack-sorting, set partitions, and Lassalle's sequence. *J. Combin. Theory Ser. A*, **175** (2020).

[3] C. Defant and J. Propp, Quantifying noninvertibility in discrete dynamical systems. *Electron. J. Combin.*, **27** (2020).