A STATISTIC FOR REGIONS OF BRAID DEFORMATIONS

Priyavrat Deshpande and Krishna Menon

Chennai Mathematical Institute, India.

Hyperplane Arrangements

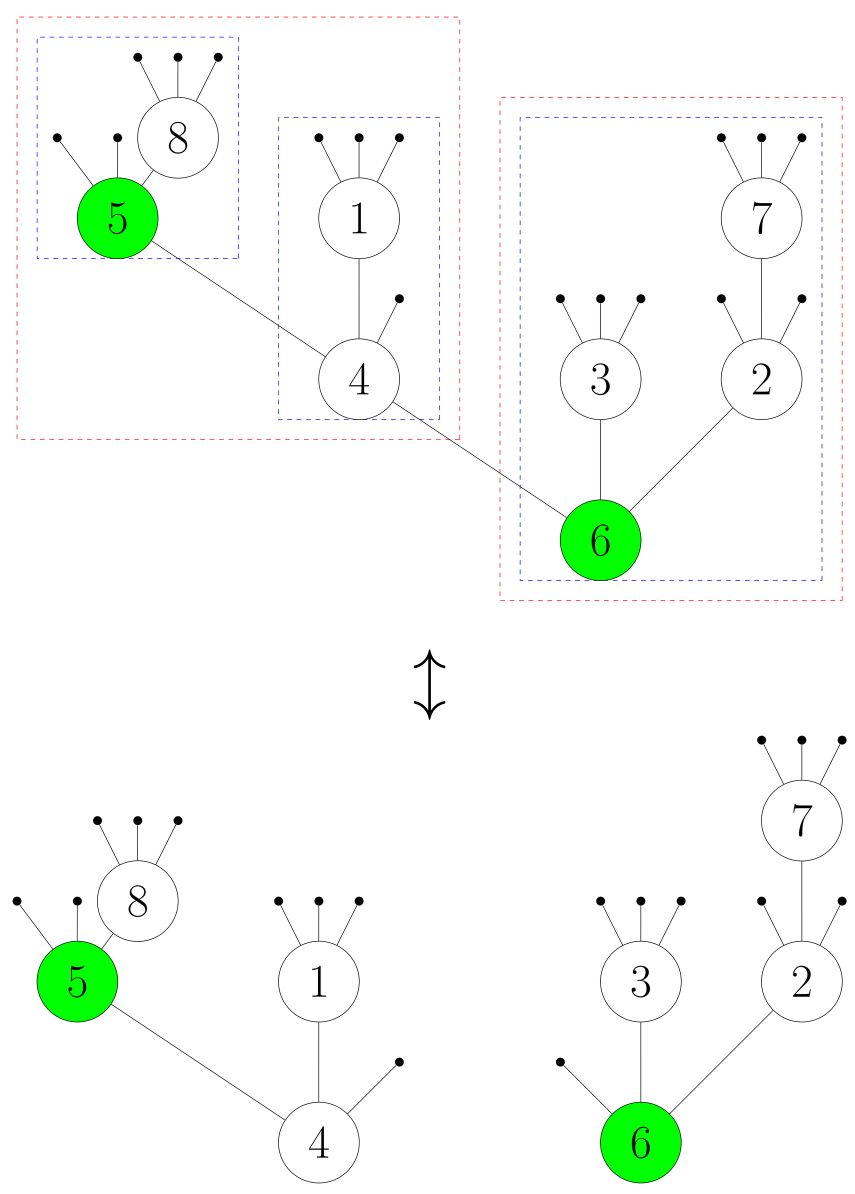
1. A hyperplane arrangement is a finite set $\mathcal{A}$ of affine hyperplanes in $\mathbb{R}^n$.
2. A region is a connected component of $\mathbb{R}^n \setminus \bigcup_{H\in\mathcal{A}} H$. The number of regions is denoted by $r(\mathcal{A})$.
3. The characteristic polynomial of $\mathcal{A}$
$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^{n-i} c_i t^i$$
encodes the combinatorics of $\mathcal{A}$.

Theorem (Zaslavsky, 1975)

$$r(\mathcal{A}) = \sum_{i=0}^{n} c_i.$$

Branch Statistic

1. The left-most nodes of a tree form its *trunk*.
2. The trunk nodes break the tree into *twigs*.
3. The trunk nodes greater than all trunk nodes after it are called *branch nodes*.
4. The branch nodes group twigs into *branches*.



Arrangements of Interest

1. Let S be a finite set of integers such that
   - $s, t \notin S$, $st > 0 \Rightarrow s + t \notin S$.
   - $s, t \notin S$, $s > 0$, $t \leq 0 \Rightarrow s - t \notin S$, $t - s \notin S$.

   Let $m = \max\{|s| : s \in S\}$.
2. The arrangement $\mathcal{A}_S(n)$ in $\mathbb{R}^n$ is given by
$$\{x_i - x_j = k \mid k \in S,\ 1 \leq i < j \leq n\}.$$
3. We have
$$\sum_{n\geq 0} \chi_{\mathcal{A}_S(n)}(t)\frac{x^n}{n!} = \left(\sum_{n\geq 0} (-1)^n r(\mathcal{A}_S(n))\frac{x^n}{n!}\right)^{-t}.$$

Objective

Find a statistic on $\mathcal{T}_S(n)$ whose distribution is given by the coefficients of $\chi_{\mathcal{A}_S(n)}(t)$.

Main Result

The coefficient c_j is the number of trees in $\mathcal{T}_S(n)$ with j branches.

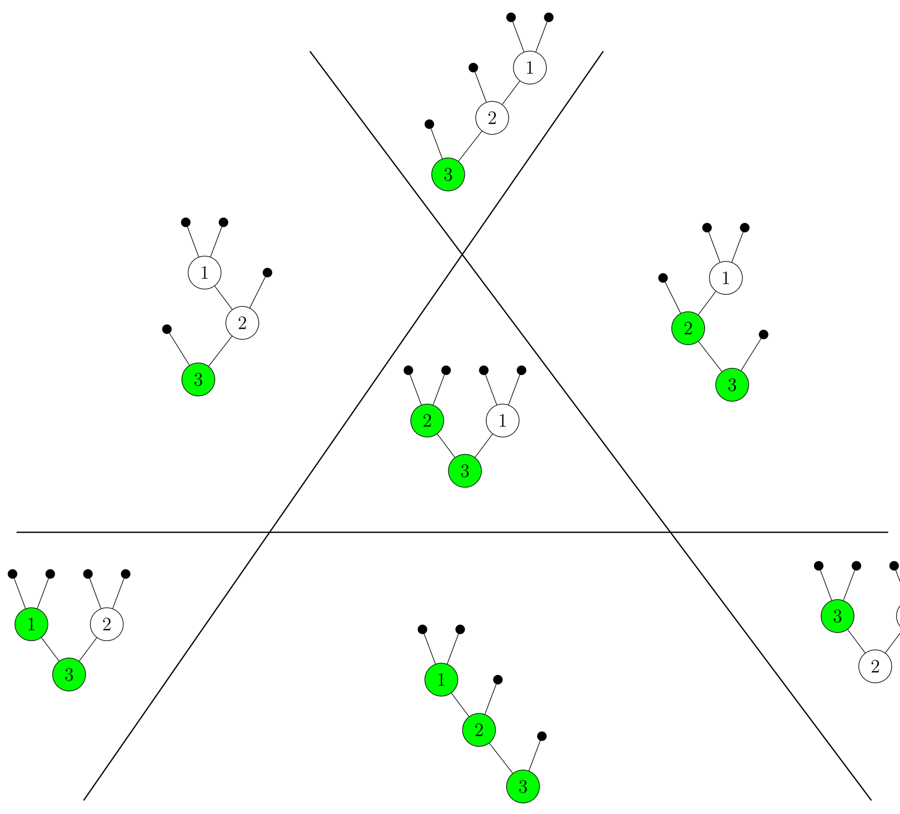


Figure: The characteristic polynomial of the Linial arrangement in $\mathbb{R}^3$ is $t^3 - 3t^2 + 3t$.

Background

The set $\mathcal{T}_S(n)$ consists of labeled $(m+1)$-ary trees with n nodes such that if $\mathrm{cadet}(i) = j$:

- $\mathrm{lsib}(j) \notin S \cup \{0\} \Rightarrow i < j$.
- $-\mathrm{lsib}(j) \notin S \Rightarrow i > j$.

Cadet of a node is its rightmost node child and $\mathrm{lsib}(j)$ is the number of left-siblings of the node j.

Theorem (Bernardi, 2016)

The regions of $\mathcal{A}_S(n)$ are in bijection with the trees in $\mathcal{T}_S(n)$.

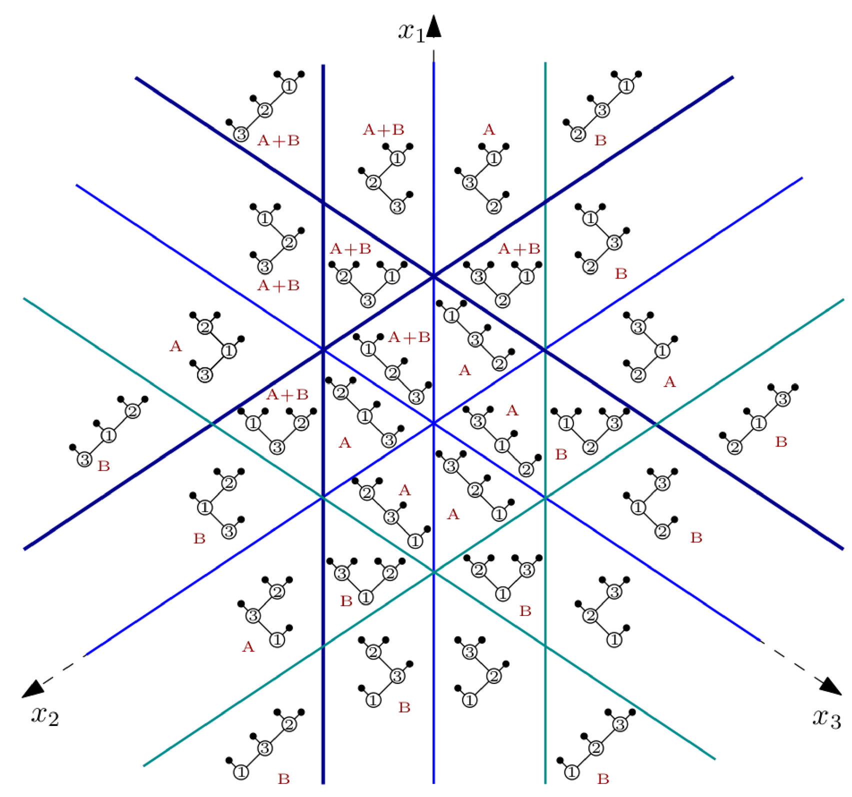


Figure: (Bernardi, 2016) Trees corresponding to regions of the Catalan arrangement in $\mathbb{R}^3$.

Exponential Structures

Given $c : \mathbb{N} \to \mathbb{N}$ for each $n, j \in \mathbb{N}$, we define
$$c_j(n) = \sum_{\{B_1,\ldots,B_j\}\in\Pi_n} c(|B_1|)\cdots c(|B_j|)$$
$$h(n) = \sum_{j=0}^{n} c_j(n).$$
In such a situation,
$$\sum_{n,j\geq 0} c_j(n)t^j\frac{x^n}{n!} = \left(\sum_{n\geq 0} h(n)\frac{x^n}{n!}\right)^t.$$
Interpreted as:

$h(n) =$ # Structures on $[n]$

$c(n) =$ # Connected structures on $[n]$

$c_j(n) =$ # Structures on $[n]$ with j components

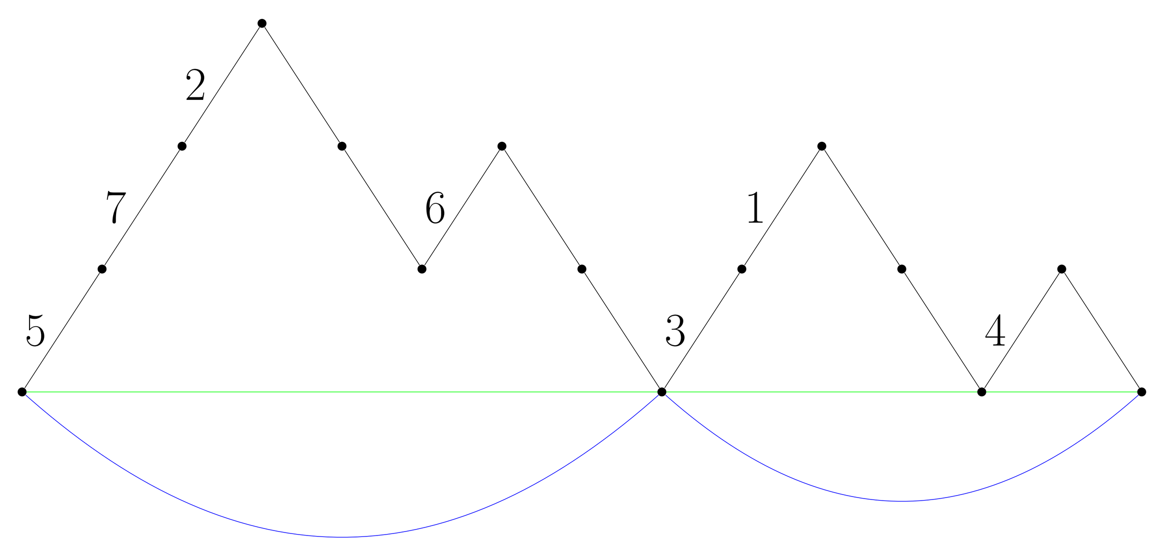
The extended Catalan arrangement

Here $S = \{-m, \ldots, m\}$.

Coefficient of t^j
$$C(m,n,j) = \sum_{k=j}^{n} (-1)^{k-j}\frac{(n-1)!}{(k-1)!}\binom{(m+1)n}{n-k} c(k,j)$$
where $c(k,j)$ are unsigned Stirling numbers of the first kind.

- $r(\mathcal{A}_S) = \frac{n!}{mn+1}\binom{(m+1)n}{n}$, *i.e.*, Fuss-Catalan numbers $\times\ n!$.
- $C(m,n,j) \leq C(m+1,n,j)$.
- $C(m,n,j) \leq C(m,n+1,j)$.
- $C(m,n,j) \geq C(m,n,j+1)$.

Similar statistics can be defined for other Catalan objects.



References

[1] O. Bernardi. Deformations of the braid arrangement and trees. *Adv. Math.*, 335:466–518, 2018.

[2] P. Deshpande and K. Menon. A branch statistic for trees: Interpreting coefficients of the characteristic polynomial of braid deformations, 2021.

[3] R. P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*. AMS, 2007.

[4] T. Zaslavsky. *Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes*, volume 154. AMS, 1975.

Contact Information

Priyavrat Deshpande

pdeshpande@cmi.ac.in

CHENNAI MATHEMATICAL INSTITUTE