Horizontal-strip LLT polynomials

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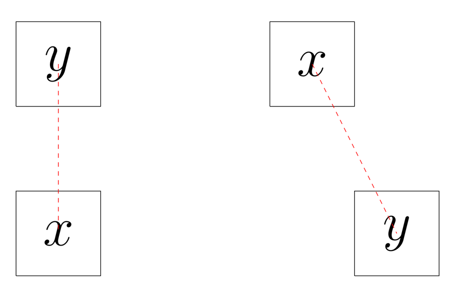
Background

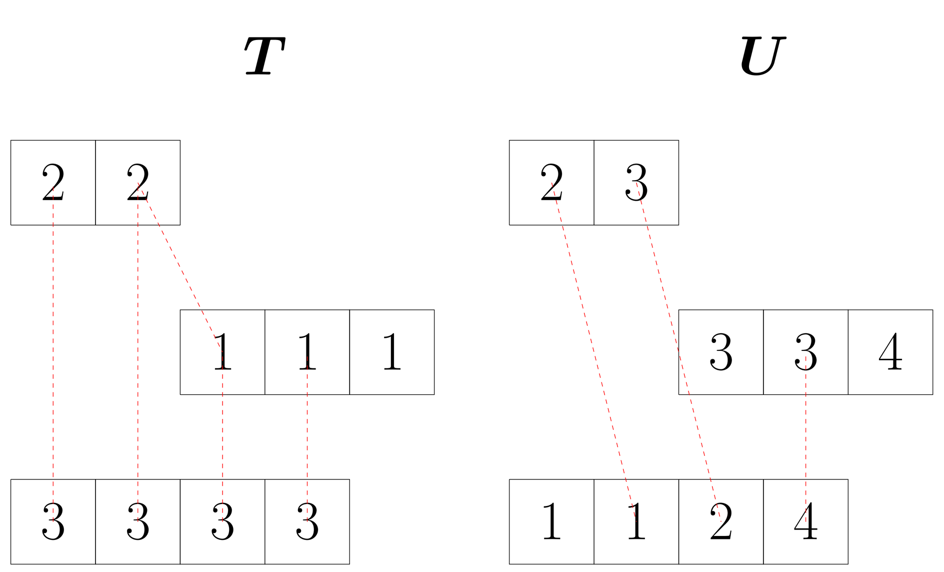
$\boldsymbol{\lambda}$: sequence of rows
$\boldsymbol{T}$: fill with weakly increasing positive integers
$\boldsymbol{x^T}$: monomial $x_1^{\text{number of 1's}} x_2^{\text{number of 2's}} \cdots$
$\text{inv}(\boldsymbol{T})$: number of pairs below with $x > y$



Horizontal-strip LLT polynomial:

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \sum_{\boldsymbol{T} \in \text{SSYT}_{\boldsymbol{\lambda}}} q^{\text{inv}(\boldsymbol{T})} \boldsymbol{x^T}$$

Example:



$$\begin{aligned} G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) &= \cdots + q^5 x_1^3 x_2^2 x_3^4 + \cdots + q^3 x_1^2 x_2^2 x_3^2 x_4^2 + \cdots \\ &= q^5 s_{432} + q^5 s_{441} + q^5 s_{522} + (q^5 + q^4) s_{531} \\ &+ 2q^4 s_{54} + 2q^4 s_{621} + (q^4 + 2q^3) s_{63} + q^3 s_{711} \\ &+ (2q^3 + q^2) s_{72} + (q^2 + q) s_{81} + s_9. \end{aligned}$$

Theorem (Lascoux, Leclerc, Thibon 1997): $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is a symmetric function.

Theorem (Grojnowski, Haiman 2007): $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q)$ is Schur-positive.

Theorem (Grojnowski, Haiman 2007): If the rows of $\boldsymbol{\lambda}$ are nested, then

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \tilde{H}_{\lambda}(\boldsymbol{x}; q) = \sum_{T \in \text{SSYT}(\lambda)} q^{\text{cocharge}(T)} s_{\text{shape}(T)}.$$

Theorem (Carlsson, Mellit 2005): We have

$$\frac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)]; q)}{(q-1)^n} = X_{\Gamma(\boldsymbol{\lambda})}(\boldsymbol{x}; q),$$

the chromatic quasisymmetric function of a certain graph $\Gamma(\boldsymbol{\lambda})$.

Main results

Definition For rows R and R', define the integer

$$M(R, R') = \begin{cases} |R \cap R'| & \text{if } R \text{ starts weakly left of } R', \\ |R \cap R'^{+}| & \text{if } R \text{ starts strictly right of } R', \end{cases}$$

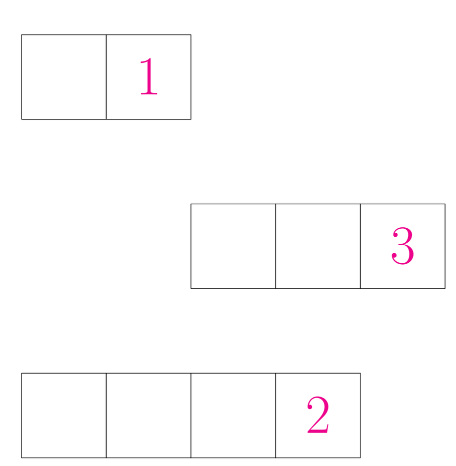
where R'^{+} is the row R' moved to the right one unit.

Definition: Weighted graph $\Pi(\boldsymbol{\lambda})$ associated to $\boldsymbol{\lambda}$

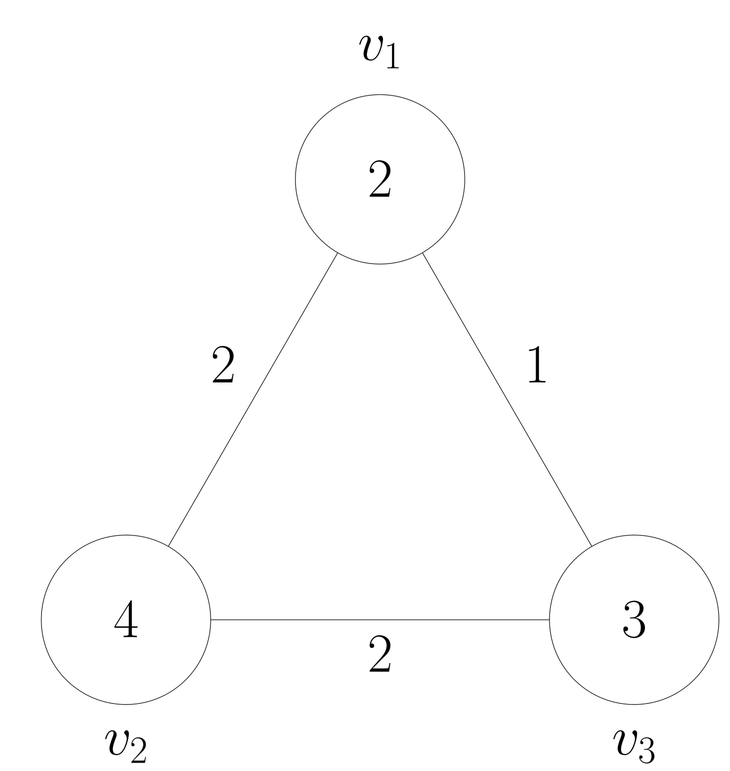
- **Vertices:** rows of $\boldsymbol{\lambda}$
- **Edges:** join attacking rows
- **Weight:** number of cells in row
- **Weight:** $M(R, R')$

Example

$\boldsymbol{\lambda} =$



$\Pi(\boldsymbol{\lambda}) =$



Theorem: LLT determined by weighted graph

If $\Pi(\boldsymbol{\lambda}) \cong \Pi(\boldsymbol{\mu})$, then $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = G_{\boldsymbol{\mu}}(\boldsymbol{x}; q)$.

Theorem: A combinatorial Schur expansion

If $\Pi(\boldsymbol{\lambda})$ is a tree, then $G_{\boldsymbol{\lambda}}(\boldsymbol{x}; q) = \sum_{T \in \text{SSYT}(\alpha)} q^{\text{cocharge}_{\Pi}(T)} s_{\text{shape}(T)}$

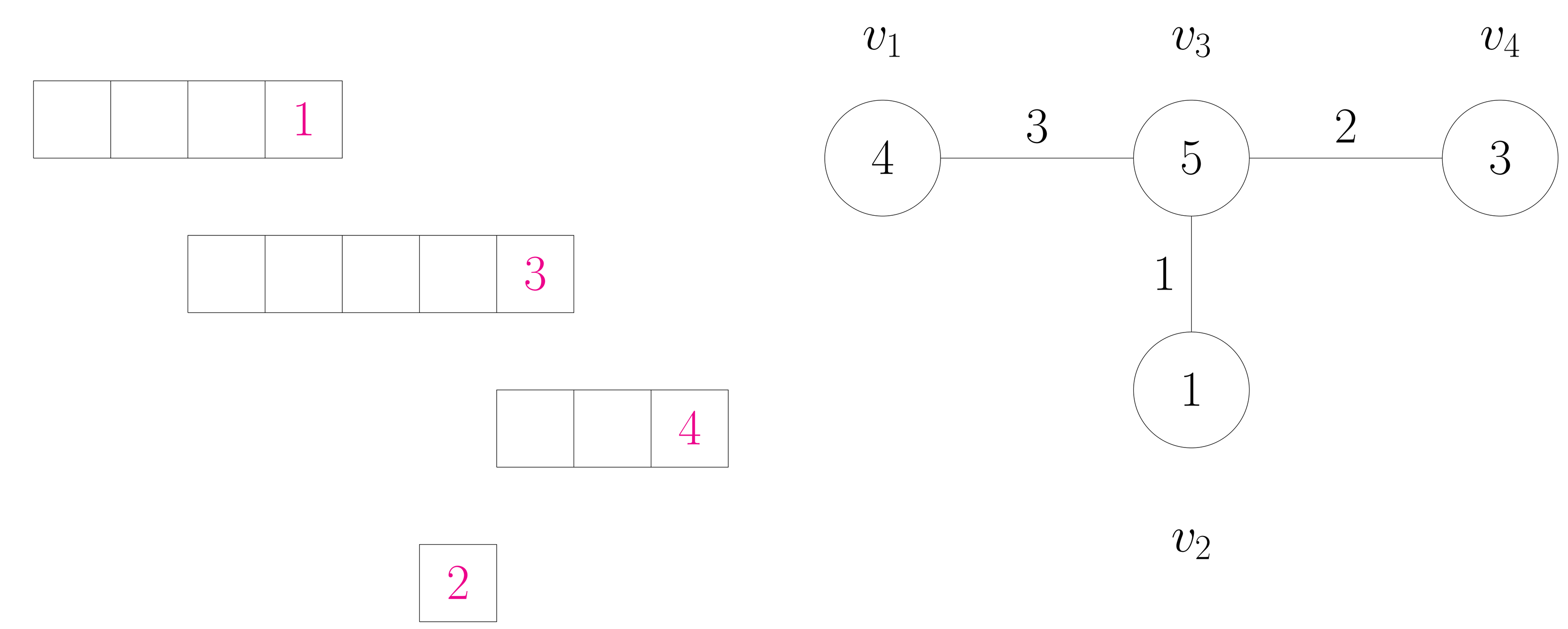
for a certain statistic $\text{cocharge}_{\Pi}(T)$ on tableaux.

Theorem: Plethystic relationship

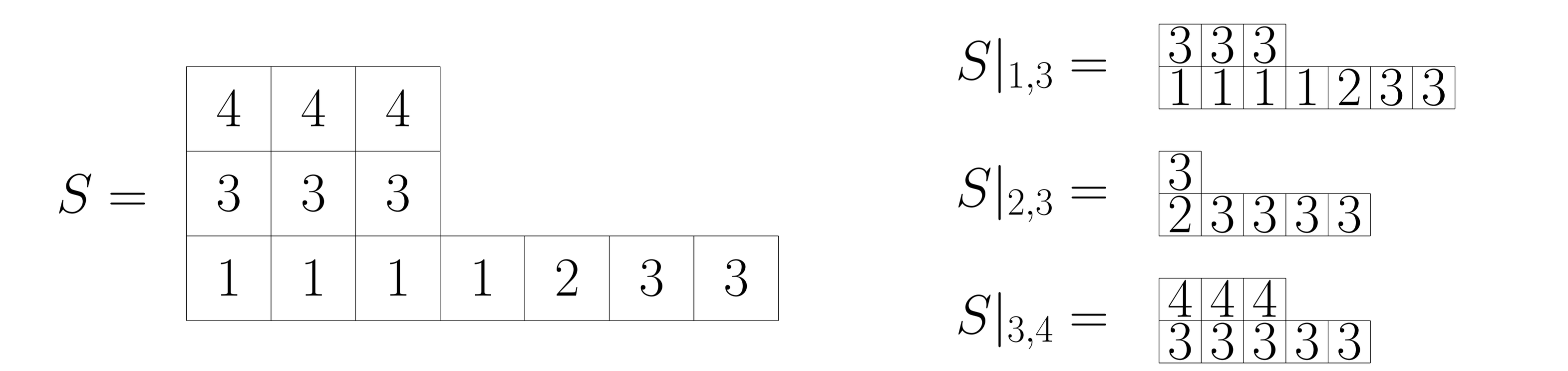
We have $\left(\dfrac{G_{\boldsymbol{\lambda}}([\boldsymbol{x}(q-1)]; q)}{(q-1)^n} \right) \Bigg|_{q=1} = X_{\Pi(\boldsymbol{\lambda})}(\boldsymbol{x}),$

the extended chromatic symmetric function of the weighted graph $\Pi(\boldsymbol{\lambda})$.

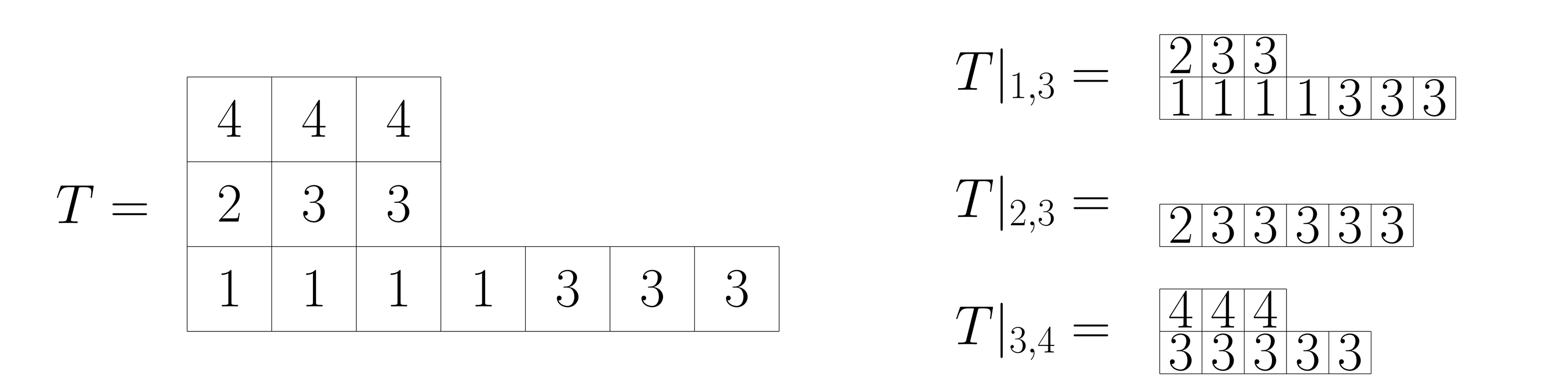
Example of combinatorial formula



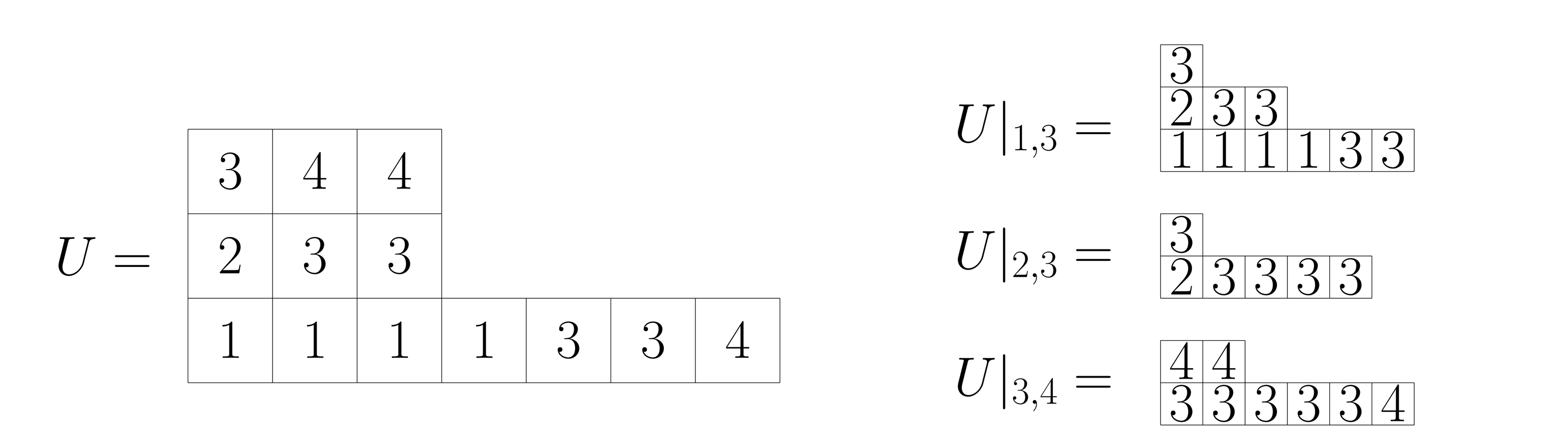
Coefficient of s_{733}: $(2q^5 + q^6)$



$\text{cocharge}_{\Pi}(S) = 2 + 1 + 2 = 5$



$\text{cocharge}_{\Pi}(T) = 3 + 0 + 2 = 5$



$\text{cocharge}_{\Pi}(U) = 3 + 1 + 2 = 6$

References

[1] I. Grojnowski and M. Haiman, *Affine Hecke algebras and positivity of LLT and Macdonald polynomials.* (2007).

[2] A. Lascoux, B. Leclerc, and J-Y. Thibon, *Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras, and unipotent varieties.* J. Math. Phys. 38 (1997), no. 2, 1041–1068.

[3] F. Tom, *A combinatorial Schur expansion of triangle-free horizontal-strip LLT polynomials.* Comb. Theory 1 14 arXiv:2011.13671 (2021).

[4] F. Tom, *A horizontal-strip LLT polynomial is determined by its weighted graph.* arXiv:2110.07984 (2021).