Asymptotics of coefficients of algebraic series via embedding into rational series

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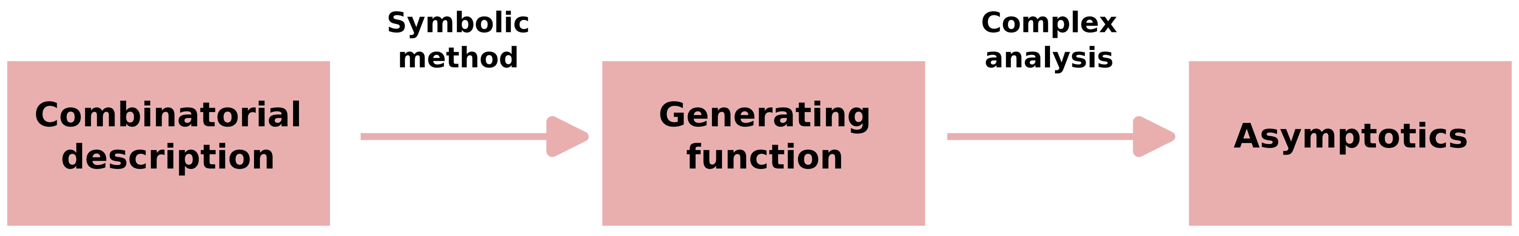
Automating counting

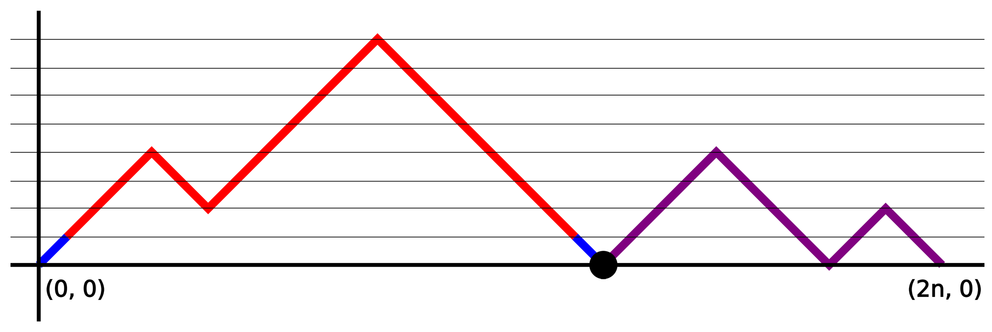
A goal in analytic combinatorics is to find asymptotics for arrays of numbers.



Big Question: Given a combinatorial description of an array, can we automate finding its asymptotics?

Example: Dyck paths

A Dyck path of size n starts at $(0,0)$, ends at $(2n,0)$, and takes steps $\{\nearrow,\searrow\}$ such that it never goes below the x-axis. Dyck paths can be uniquely decomposed according to their first return to the x-axis:



Dyck path = 2 legs + arbitrary Dyck path + arbitrary Dyck path

$$\text{Symbolic method} \rightarrow D(z) = 1 + z\cdot D(z)\cdot D(z) = \frac{1-\sqrt{1-4z}}{2z}$$

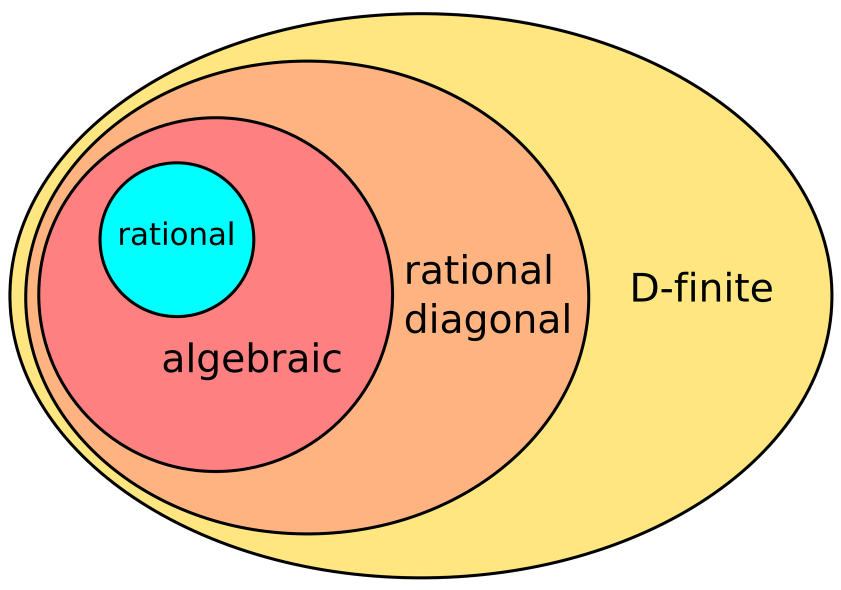
We could use Taylor series to derive a formula for the Catalan numbers, or for asymptotics, note that the singularity with smallest modulus is at $z = 1/4$.

Context

- Flajolet and Sedgewick's book, [3]: univariate generating functions.
- Pemantle and Wilson's book, [5]: multivariate rational generating functions $F(\mathbf{x}) = \sum_{n_1,\ldots,n_d} a_{i_1,\ldots,i_d} x_1^{n_1}\cdots x_d^{n_d}$. Here, $\mathbf{x} = (x_1,\ldots,x_d)$.
- Analytic combinatorics mantra:
  - Location of a GF's singularities determines exponential growth of its coefficients.
  - Behavior of the GF near its singularities determines subexponential growth.
- The Cauchy integral formula is central to these derivations:

$$\left[x_1^{n_1}\cdots x_d^{n_d}\right] F(\mathbf{x}) = \left(\frac{1}{2\pi i}\right)^d \int_T \frac{F(\mathbf{x})}{x_1^{n_1+1}\cdots x_d^{n_d+1}}\, dx_1\cdots dx_d$$

Hierarchy of GFs



- Rational GFs encode the output of deterministic finite automata.
- Algebraic GFs encode outputs of the more expressive context-free grammars. Examples include Dyck paths, binary trees, constrained (random) walks, and RNA secondary structures.

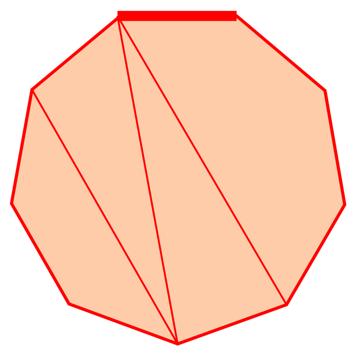
Definition. The *elementary diagonal* of a GF $F(\mathbf{x})$ is the $(d-1)$-variate GF encoding the coefficients from F where x_1 and x_2 have matching powers.

Embedding with Safonov

The following from [6] is a generalization of a result from Furstenberg, [4]. Suppose that f is an algebraic power series given as a branch of $P(f(x), x) = 0$, that f is divisible by x_1, and that in some neighborhood of $\mathbf{0}$, there is a factorization $P(Y,x) = (Y - f(x))^k u(Y,x)$ where $u(0,\mathbf{0}) \neq 0$ and $k \geq 1$ is an integer. Then f is the elementary diagonal of the rational function F given by

$$F(Y,x) = \frac{Y^2 P_Y(Y, Yx_1, x_2, \ldots, x_d)}{kP(Y, Yx_1, x_2, \ldots, x_d)}.$$

Example: Dissections



Drmota [1, p.376] enumerates dissections of polygons using a bivariate GF $A(x,y)$, where x counts the number of vertices in the polygon, and y counts the total number of edges in the dissection. A satisfies:

$$A(x,y) = xy^2(1+A)^2 + xy(1+A)\cdot A.$$

Since A is divisible by x, we embed into

$$F(Y,x,y) = \frac{\left(1 - \left(2Y^2xy^2 + 2Y^2xy + 2Yxy^2 + Yxy\right)\right)Y}{1 - \left(Y^2xy^2 + Y^2xy + 2Yxy^2 + Yxy + xy^2\right)}.$$

We note that $[x^{pn}y^{(1-p)n}]A(x,y) = [Y^{pn}x^{pn}y^{(1-p)n}]F(Y,x,y)$, and find that there is a single smooth critical point in this direction. Thus for $1/3 < p < 1/2$:

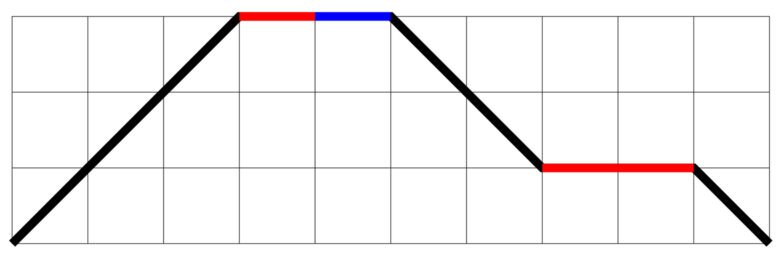
$$\left[x^{pn}y^{(1-p)n}\right] A(x,y) = \frac{\sqrt{1-p}}{2\pi p^2\sqrt{3p-1}}\cdot\frac{1}{n^2}\left(\frac{(1-p)^{1-p}}{(1-2p)^{2-4p}(3p-1)^{3p-1}}\right)^n + O\left(\frac{1}{n^3}\right).$$

Preprocessing methods

Preprocessing may lead to Safonov's conditions or nicer embeddings.

- If $F(\mathbf{0}) \neq 0$, subtract off constant or polynomial.
- If F not divisible by a variable, sub $x \to xy$.
- Sometimes, a substitution $F \to F/x$ simplifies embeddings.

Example: Bicolored Motzkin paths



In [2, Lemma 2.1], Elizalde derives the GF for bicolored Motzkin paths with steps $\{\nearrow,\searrow,\rightarrow,\rightarrow\}$. Let $a_{m,n}$ be the number of such paths with m total $\nearrow$ or $\rightarrow$ steps and n total $\searrow$ or $\rightarrow$ steps. Then

$$M(x,y) := \sum_{m,n=0}^{\infty} a_{m,n}x^m y^n = \frac{1-x-y-\sqrt{(1-x-y)^2-4xy}}{2xy}.$$

Using Safonov on $M(x,xy) - 1$ gives the asymptotic formula

$$\left[x^{pn}y^{(1-p)n}\right] M(x,y) = \frac{1}{2\pi(1-p)^2p^2}\cdot\frac{1}{n^2}\cdot\left(\frac{(1-p)^{2p-2}}{p^{2p}}\right)^n + O\left(\frac{1}{n^3}\right).$$

Future work

- How does Denef-Lipshitz, another embedding strategy, compare?
- Can we find embeddings for GFs satisfying quadratics/cubics/quartics?
- Can we systematize generating "nice" embeddings?

arXiv & acknowledgements

Our online database of 20 algebraic examples from the literature is available online, with Sage code illustrating these embeddings. This work is supported by the NSF under Grant Numbers 1641020 and 1916439, an NSERC Discovery Grant, an NSERC USRA and the AMS MRC, Combinatorial Applications of Computational Geometry and Algebraic Topology.

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