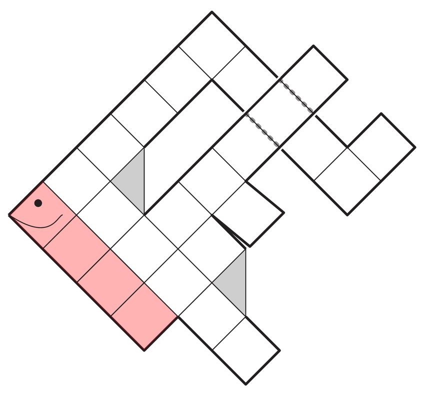

Bijections between Fighting Fish, Planar Maps and Tamari Intervals

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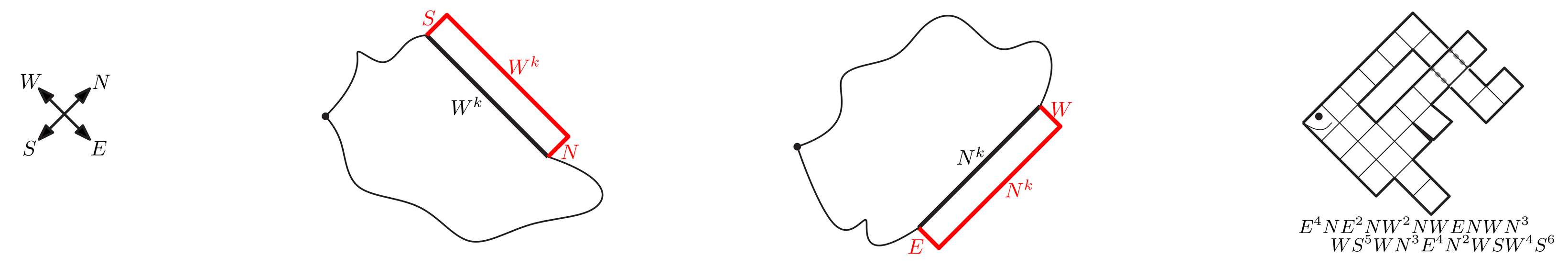
Fighting Fish

We define 2 operations on finite words on $\Sigma = \{E, N, W, S\}$:

- operation $\triangledown_k$, $k \geq 0$: replace a subword N^k by EN^kW.
- operation $\triangle_\ell$, $\ell \geq 0$: replace a subword W^k by NW^kS.



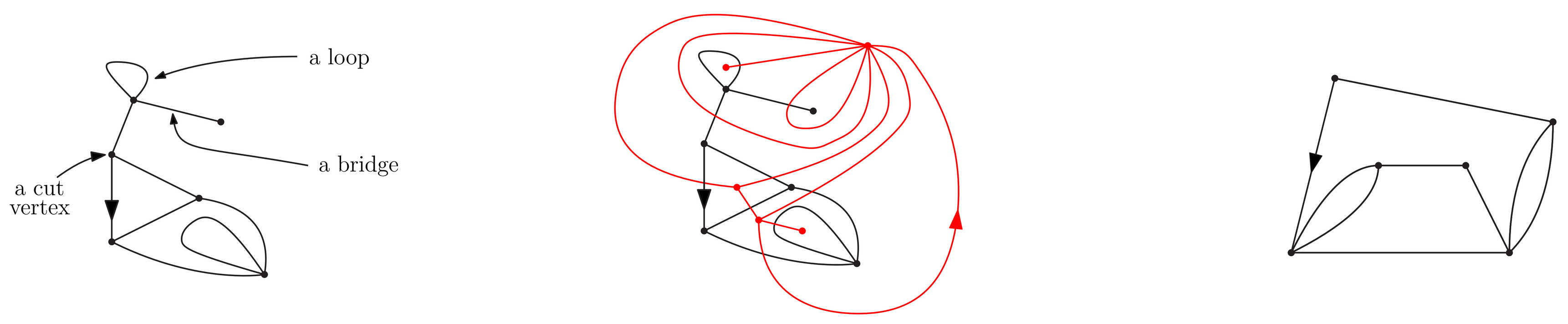
A **fighting fish** is a word obtainable from the word $ENWS$ using operations $\triangledown_k$ and $\triangle_\ell$ for $k, \ell \geq 1$. Its **size** is its semilength ($= \#E + \#N = \#W + \#S$).
We denote by $\mathcal{FF}_n$ the set of fighting fish of size n. See [1] for an introduction/review.
Enumerated by $|\mathcal{FF}_{n+1}| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}$ (starting by 1,2,6,22,91,408,...) : same sequence as *nonseparable planar maps*, *synchronized intervals of the Tamari lattice*, two-stack sortable permutations, left ternary trees, ...

Generalized Fighting Fish

A **generalized fighting fish** is a word obtainable from the empy word using operations $\triangledown_k$ and $\triangle_\ell$ for $k, \ell \geq 0$. Its **size** is its semilength.
We denote by $\mathcal{GFF}_n$ the set of generalized fighting fish of size n. Note that $\mathcal{FF} \subseteq \mathcal{GFF}$.
A **down bridge** (resp. **up bridge**) of $F \in \mathcal{GFF}$ is a decomposition $F = F_1EGWF_2$ (resp; $F = F_1NGSF_2$) such that G and F_1F_2 are generalized fighting fish.

Planar maps

A **planar map** is a proper embedding of a connected multigraph on the plane, defined up to continuous deformations. A planar map splits the plane into **edges**, **vertices** and **faces**. We will always consider planar maps as **rooted** : an edge (the *root edge*) incident to the outer face (the *root face*) is distinguished and oriented towards a vertex (the *root vertex*) such that the outer face is on its right. A **nonseparable planar map** is a planar map without **cut vertices**, i.e. vertices whose deletion would disconnect the map. We denote by $\mathcal{M}_n$ (resp. $\mathcal{NSM}_n$) the set of planar maps (resp. nonseparable planar maps) with n edges.

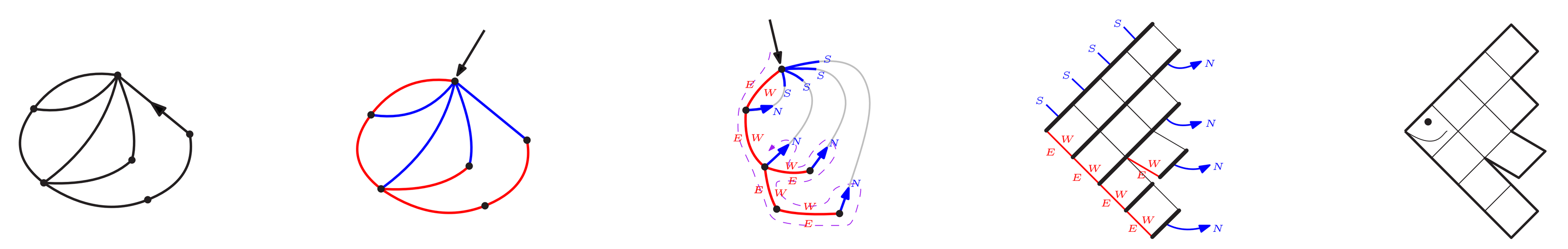


A **loop** is an edge with both ends incident to the same vertex. A **bridge** is an edge whose deletion would disconnect the map. The **dual** of a rooted planar map M is the map $\overline{M}$ whose vertices are faces of M, whose edges are the duals of edges of M (linking adjacent faces of M), rooted in such a way that the root face (resp. vertex) of M becomes the root vertex (resp. face) of $\overline{M}$.

The Mullin encoding of a planar map

For a planar map M, its Mullin encoding $\Phi(M)$ is the word obtained via the following procedure :

1. Endow M with its rightmost depth-first search spanning tree T.
2. Explore the map M with a counterclockwise traversal of T, register a E (resp. a W) if we go along an edge of T for the first (resp. second) time and register a N (resp. S) if we cross an edge not in T for the first (resp. second) time.



Theorem [2] : Φ is a bijection between $\mathcal{M}$ and $\mathcal{GFF}$ and between $\mathcal{NSM}$ and $\mathcal{FF}$, with the following statistics correspondence :

$\mathcal{M}$	#edges	#vertices	#faces	#loops	#bridges
$\mathcal{GFF}$	size	$\#E+1$	$\#N+1$	#up bridges	#down bridges

Also, Φ preserves duality : $\overline{\Phi(M)} = \Phi(\overline{M})$.
Counting sequence : $|\mathcal{GFF}_n| = |\mathcal{M}_n| = \frac{2.3^n}{(n+1)(n+2)}\binom{2n}{n}$ (starting by 1,2,9,54,378,2916,...)

Some perspectives :

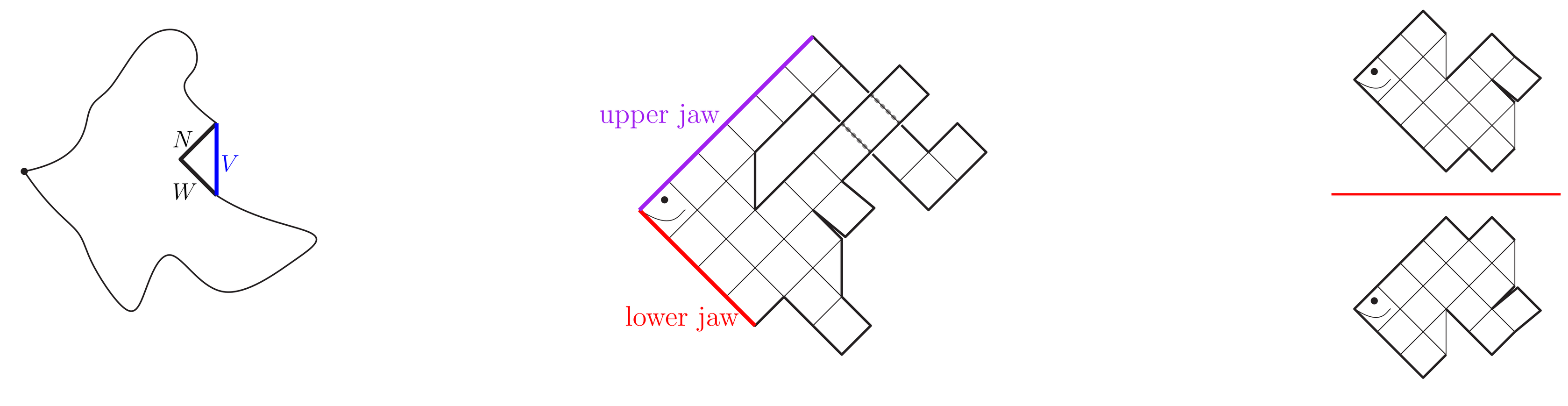
- Find a natural master model of fighting fish unifying generalized and extended fighting fish.
- Find bijections between fighting fish and left ternary trees, extended fighting fish and rooted simple triangulations.
- What is the fish model for m-Tamari lattices (the m-Dyck paths analogue) ?

Some references :

[1] *Fighting fish*, Duchi, Guerrini, Rinaldi, Schaeffer (2017).
[2] *Bijections between fighting fish, planar maps and Tamari intervals*, Duchi, Henriet (2022).
[3] *A bijection between Tamari intervals and extended fighting fish*, Duchi, Henriet (2022).
[4] *Higher trivariate diagonal harmonics via generalized Tamari posets*, Bergeron, Préville-Ratelle (2011).
[5] *The Rise-Contact involution on Tamari intervals*, Pons (2019).

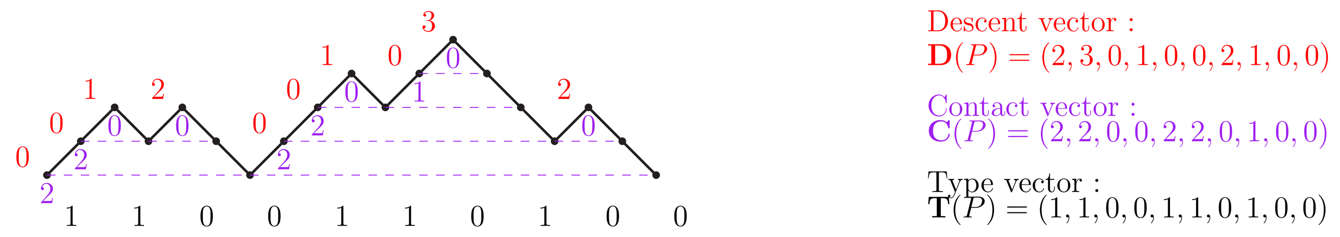
Extended Fighting Fish

An **extended fighting fish** is a word on $\{E, N, W, S, V\}$ obtainable from the word $ENWS$ using operations $\triangledown_k$ and $\triangle_\ell$ for $k, \ell \geq 1$ and the new operation $\triangleleft$ that consists in replacing a subword WN by V. Its **size** is its number of lower letters ($\#E + \#N$) : V letters are considered as "free steps".
We denote by $\mathcal{EFF}_n$ the set of extended fighting fish of size n, we have $\mathcal{FF}_n \subseteq \mathcal{EFF}_n$.



The **lower jaw** (resp. **upper jaw**) of $F \in \mathcal{EFF}$ is the maximal integer k such that E^k is a prefix of F (resp. S^k is a suffix of F). The **area** of an extended fighting fish is the number of full squares it contains.
The **conjugate** of $F \in \mathcal{EFF}$ is the extended fighting fish $\overline{F}$ obtained by reversing F and changing the letters with the rules $E \leftrightarrow S$, $N \leftrightarrow W$.

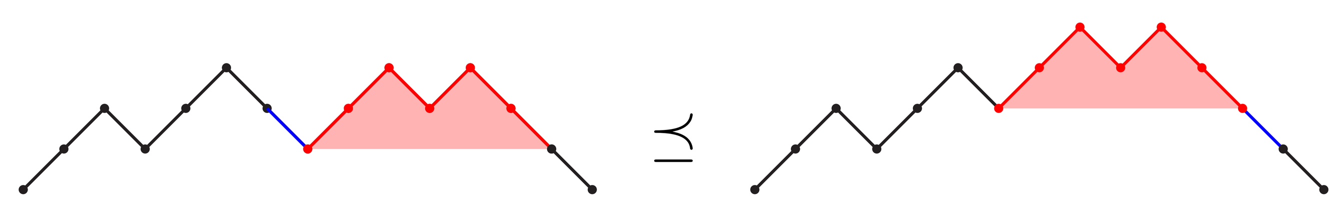
Intervals of the Tamari lattice



A **Dyck path** of **size** n, or n-Dyck path, is a finite walk from $(0, 0)$ to $(2n, 0)$ staying weakly below the x-axis, with n up steps $u = (1, 1)$ and n down steps $d = (1, -1)$.
For a Dyck path P, its **last descent** is the number of down steps it ends with, and its **number of contacts** is the number of its down steps ending on the x-axis.

The conjugate of a Dyck path is defined inductively : $\begin{cases} \overline{\bullet} = \bullet \\ \overline{P_1uP_2d} = \overline{P_2}u\overline{P_1}d \end{cases}$.

The **Tamari lattice** $\mathcal{D}_n$ is the set of Dyck paths of size n endowed with the partial order $\preceq$ given by the reflexive and transitive closure of the **right rotation** :



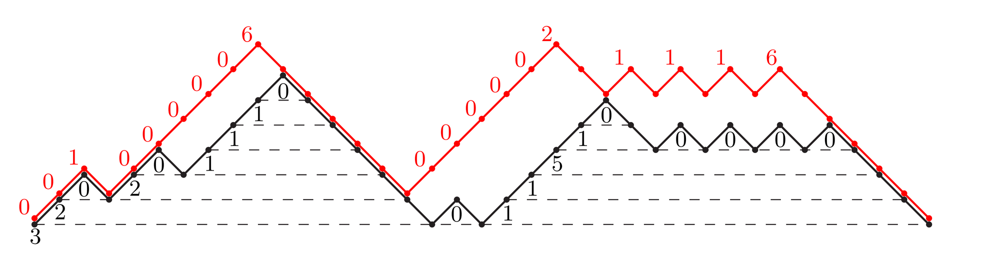
A **Tamari interval** of size n is a pair of n-Dyck paths $[P, Q]$ with $P \preceq Q$.
A Tamari interval $[P, Q]$ is **synchronized** if $\mathbf{T}(P) = \mathbf{T}(Q)$.
We denote by $\mathcal{I}_n$ (resp. $\mathcal{SI}_n$) the set of Tamari intervals (resp. synchronized intervals) of size n.
For a Tamari interval $I = [P, Q]$ its **last descent** is the last descent of Q, its **number of contacts** is the number of contacts of P, and its **Tamari distance** is the length of the longest strictly increasing chain from P to Q in the Tamari lattice.
The **conjugate** of a Tamari interval $I = [P, Q]$ is $\overline{I} = [\overline{Q}, \overline{P}]$.

Bijection between $\mathcal{I}$ and $\mathcal{EFF}$

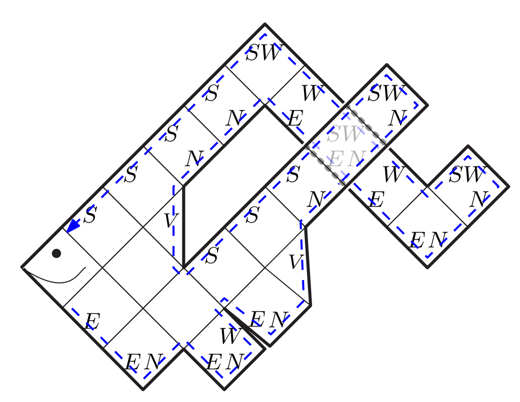
Let $I = [P, Q]$ be a Tamari interval of size n, with $\mathbf{C}(I) = (c_0, ..., c_n)$ and $\mathbf{D}(I) = (d_0, ..., d_n)$ its contact and descent vectors. For $0 \leq i \leq n$, we set :

$$\begin{aligned} w_i &= E^{c_i(P)-1}N && \text{if } c_i(P) \geq 1 \text{ and } d_{n-i}(Q) = 0 \\ w_i &= WS^{d_{n-i}(Q)-1} && \text{if } c_i(P) = 0 \text{ and } d_{n-i}(Q) \geq 1 \\ w_i &= V && \text{if } c_i(P) = 0 \text{ and } d_{n-i}(Q) = 0 \end{aligned}$$

We define then $\Psi(I) = Ew_0w_1...w_nS$.



$\mathbf{C}(I) = (3, 2, 0, 2, 0, 1, 1, 1, 0, 0, 1, 1, 5, 1, 0, 0, 0, 0, 0)$
$\overleftarrow{\mathbf{D}}(I) = (0, 0, 1, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 2, 1, 1, 1, 6)$
$E^2N\ EN\ W\ EN\ V\ N\ N\ N\ WS^5\ V\ N\ N\ E^4N\ N\ WS\ W\ W\ W\ WS^5$



Theorem [3] : Ψ is a bijection between $\mathcal{EFF}$ and $\mathcal{I}$ and between $\mathcal{FF}$ and $\mathcal{SI}$, with the following statistics correspondence :

$\mathcal{I}$	size	Tamari distance	#valleys of P	#valleys of Q	#double rises of P	#double rises of Q
$\mathcal{EFF}$	size + 1	area+size	$\#E-1$	$\#W-1$	$\#N-1$	$\#S-1$

Also, Ψ preserves symmetry : $\overline{\Psi(I)} = \Psi(\overline{I})$.
Counting sequence : $|\mathcal{EFF}_{n+1}| = |\mathcal{I}_n| = \frac{2}{(n+1)(3n+2)}\binom{4n+1}{n}$ (starting by 1,3,13,68,399,2530,...).

Application : a formula for Tamari distance

With the area-distance correspondence, the following formula came up naturally :
Theorem [3] : For every Tamari interval $I = [P, Q]$, its Tamari distance $d(I)$ writes :

$$d(I) = \sum_{0 \leq i < j \leq n} (c_i(P) - 1)(1 - d_{n-j}(Q))$$

See [4] for a symmetric group representation's view of this statistic, and [5] for an exploration of the numerous symmetries of the Tamari world.