Rooted Clusters for Graph LP Algebras

Esther Banaian [1] Sunita Chepuri [2] Elizabeth Kelley [3] and Sylvester W. Zhang [4]

[1]Aarhus University [2]University of Michigan [3]University of Illinois Urbana-Champaign [4]University of Minnesota

Cluster Algebras & LP Algebras

Cluster algebras are a type of commutative ring whose generators $x_1, \ldots, x_n$ appear in *clusters* $\{x_1, \ldots, x_k\}$ of fixed size. Given a cluster $\mathcal{C}$, the *mutation* operation μ_i uniquely exchanges $x_i \in \mathcal{C} = \{x_1, \ldots, x_k\}$ for some $x'_i \in \mathcal{C}' = \{x_1, \ldots, x'_i, \ldots, x_k\}$ via a binomial *exchange relation*.

Two important structural properties of cluster algebras are:

- **The Laurent Phenomenon:** Every cluster variable can be written as a Laurent polynomial in terms of any cluster.
- **Positivity:** The coefficients of this Laurent polynomial are strictly non-negative.

Laurent Phenomenon (LP) algebras are a generalization of cluster algebras defined by Lam and Pylyavskyy [2] where the binomial exchange relations are replaced by irreducible Laurent polynomials. By design, these algebras exhibit the Laurent Phenomenon. In general, positivity remains conjectural.

Graph LP Algebras

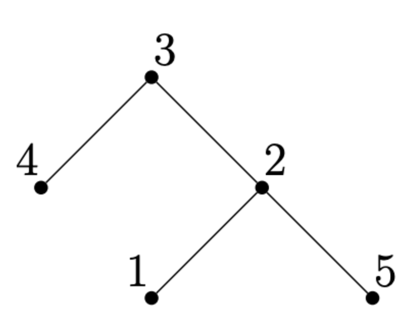
Graph LP Algebras are a subclass of LP Algebras whose generators and relations have nice combinatorial encodings in terms of graphs.

Theorem (Lam-Pylyavskyy, 2016 [3]):

Let Γ be an undirected graph on $[n]$. Define $\mathfrak{N} = [\mathfrak{n}_{ij}]$ as

$$\mathfrak{n}_{ij} = \begin{cases} \frac{1}{X_i}\left(A_i + \sum_{i \text{ adjacent to } j} X_j\right) & \text{if } i = j, \\ -1 & \text{if } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

The graph LP algebra $\mathcal{A}_\Gamma$ has cluster variables $\{X_1, \ldots, X_n\} \cup \{Y_S : S \subset [n] \text{ is connected}\}$, where $Y_S = |\mathfrak{N}_S|$. Its clusters have the form $\{X_{i_1}, \ldots, X_{i_k}\} \cup \{Y_S\}_{S \in \mathcal{S}}$ where $\mathcal{S}$ is a maximally nested collection of subsets of $[n] \backslash \{i_1, \ldots, i_k\}$.



$\mathcal{A}_\Gamma$ has cluster variables $X_1, X_2, X_3, X_4, X_5, Y_1, Y_2, Y_3, Y_4, Y_5, Y_{12}, Y_{25}, Y_{23}, Y_{34}, Y_{125}, Y_{235}, Y_{123}, Y_{234}$, and Y_{12345}.

Examples of clusters include: $\{X_3, Y_4, Y_{125}, Y_{25}, Y_5\}$, $\{X_1, X_2, X_3, X_4, X_5\}$, and $\{Y_4, Y_2, Y_{12}, Y_{125}, Y_{12345}\}$.

Figure 1. An example of a graph, Γ, and several clusters in the corresponding algebra $\mathcal{A}_\Gamma$.

Rooted Clusters

Definition:

Let Γ be a tree on $[n]$ and choose a vertex, v, to be its root. For each vertex x in Γ, let $I_x := \{x\} \cup \{\text{descendants of } x\}$. The *rooted cluster* associated to v is $C_v := \{I_x\}_{x \in [n]}$.

For Γ from Figure 1, an example of a rooted cluster is $\mathcal{C}_3 = \{Y_1, Y_4, Y_5, Y_{125}, Y_{12345}\}$.

Explicit Formulas

We study the case where Γ is a tree with vertex set $[n]$ and $\mathcal{C}$ is a rooted cluster on Γ.

Theorem (BCKZ, 2021):

For any $i \in V(\Gamma)$, X_i can be expressed as a Laurent polynomial with non-negative coefficients in the elements of $\mathcal{C}$. Similarly, for any $S \subset V(\Gamma)$, Y_S can be expressed as a Laurent polynomial with non-negative coefficients in the elements of $\mathcal{C}$. Explicitly,

$$X_i = \frac{\sum_{u \in \Gamma^v_{\geq i}} \left(\prod_{w \in \Gamma^v_{\geq i} \backslash \Gamma^v_{\geq u}} Y_{\Gamma^v_{<w}}\right) \left(\prod_{w \in \Gamma^v_{<u}} Y_{I_w}\right) \left(\sum_{w \in I_u} Y_{\Gamma^v_{<u} \backslash \Gamma^v_{\geq w}} A_w\right)}{\prod_{u \in \Gamma^v_{\geq i}} Y_{I_u}}$$

$$Y_S = \sum_{\substack{O \subseteq S \text{ containing all} \\ \text{minimal elements of } \Gamma \text{ in } S}} \; \sum_{\substack{u: S \backslash O \to V(\Gamma) \\ u(x) \in \Gamma^v_{\lessdot x} \backslash O}} \frac{\left(\prod_{x \in O} Y_{I_x}\right) \left(\prod_{x \in S \backslash O} Y_{\Gamma^v_{<x} \backslash \{u(x)\}}\right)}{\prod_{x \in S} Y_{\Gamma^v_{<x}}}$$

Hyper T-path Construction

We extend Schiffler's T-path construction for Type A_n cluster algebras [4]. We construct an auxiliary graph $\Gamma_{\mathcal{C}}$ by adding a vertex i' adjacent to each leaf i of Γ and then, for each $S \in \mathcal{C}$, adding a hyperedge labeled Y_S that connects the neighbors of S in the extended graph.

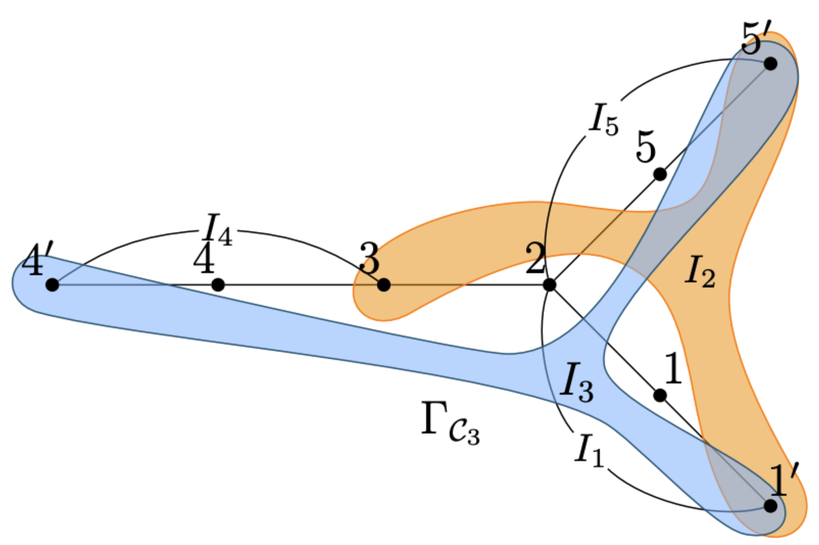


Figure 2. The auxiliary graph $\mathcal{C}_3$ for Γ from Figure 1.

Hyper T-path Expansion Formula

Theorem (BCKZ, 2021):

Let Γ be a tree and $\mathcal{C}$ a rooted cluster on Γ. If S is a connected subset of vertices of Γ, then

$$Y_S = \sum_{\substack{\text{complete hyper} \\ T\text{-paths } \alpha \text{ for } S}} \mathrm{wt}(\alpha)$$

where

$$\mathrm{wt}(\alpha) = \left(\prod_{\text{odd connections } c} \mathrm{wt}(c)\right) \left(\prod_{\text{even connections } c} wt(c)\right)^{-1}$$

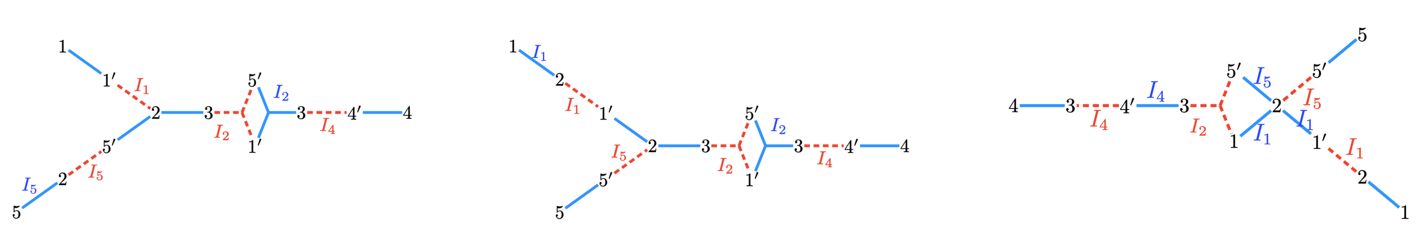
Hyper T-path Rules

Let S be a connected subset of Γ. A *complete hyper T-path* for S with respect to $\mathcal{C}$ is a set of nodes, labelled by vertices of $\Gamma_{\mathcal{C}}$, joined by connections labelled by hyperedges of $\Gamma_{\mathcal{C}}$ such that the diagram is connected and the following hold.

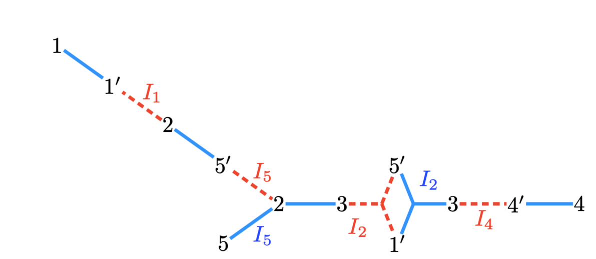
1. If a connection is labelled by hyperedge e, it joins nodes labelled by all the endpoints of e with multiplicity 1.
2. There are a distinguished set of *boundary nodes* labelled by elements of S' with multiplicity 1. Other nodes are called *internal nodes*.
3. Connections are specified to be *even* or *odd*.
4. Boundary nodes are adjacent only to odd connections.
5. Internal nodes labelled by elements of S are adjacent to exactly one even and at least one odd connection.
6. Internal nodes labelled by elements not in S are adjacent to exactly one even and exactly one odd connection.
7. If x, y are below elements of S, any path in any complete hyper T-path from boundary node x to boundary node y uses even connections labelled, in order, by $I_x, I_{a_p}, I_{a_{p-1}}, \ldots, I_{a_1}, I_{b_1}, I_{b_2}, \ldots, I_{b_q}, I_y$ where the shortest path from x to y in Γ' is $x, a_p, a_{p-1}, \ldots, a_1, x \vee y, b_1, b_2, \ldots, b_q, y$ for $p, q \geq 0$.
8. If x is below an element of S and y above the maximal element of S, any path in any complete hyper T-path from the boundary node x to the boundary node y uses even connections labelled, in order, by $I_x, I_{a_p}, \ldots, I_{a_2}$, where the shortest path from x to y in Γ' is $x, a_p, a_{p-1}, \ldots, a_1, y$, $p \geq 1$. If $p = 1$, then a path from x to y uses the even connection I_x.
9. If x, y are boundary nodes, where the shortest path from x to y in Γ' is $x, a_p, \ldots, a_1, x \vee y, b_1, \ldots, b_q, y$, then any path in any complete hyper T-path from x to y uses nodes labelled by elements of $\mathcal{L}_{x \vee y}$ and $a_p, a_{p-1}, \ldots,$ $a_1, x \vee y, b_1, b_2, \ldots, b_q$, with any multiplicity. If one of the nodes, say y, is adjacent to the maximal element of S, then $x \vee y = y$ and $q = 0$.

Examples and Non-Examples

The following are three examples of valid hyper T-paths for $S = \{2, 3\}$ with respect to $\mathcal{C}_3$.



The following is a *non-example* because it violates the ordering condition for even connections given in Rule (7).



Work In Progress

We are currently working on an analogue of *snake graphs*, a type of combinatorial object used in some proofs of positivity. Our construction already yields positivity in some additional cases.

References

[1] Sergey Fomin and Andrei Zelevinsky. Cluster algebras i: Foundations. *Journal of the American Mathematical Society*, 15(2):497–529, 2002.

[2] Thomas Lam and Pavlo Pylyavskyy. Laurent phenomenon algebras. *Cambridge Journal of Mathematics*, 4(1):121–162, 2016.

[3] Thomas Lam and Pavlo Pylyavskyy. Linear Laurent phenomenon algebras. *International Mathematics Research Notices*, 2016(10):3163–3203, 2016.

[4] Ralf Schiffler. A cluster expansion formula (a_n case). *Electronic Journal of Combinatorics*, 2008.

Acknowledgements: Sunita Chepuri was supported by NSF grant no. 1439786. Elizabeth Kelley was supported by NSF grant no. 1937241.

Contact: kelleye@illinois.edu