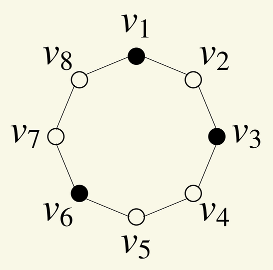
Toggling Independent Sets of a Cycle Graph

Colin Defant (Princeton → MIT), Michael Joseph (Dalton State), Matthew Macauley (Clemson), Alex McDonough* (UC Davis)

Independent Sets of a Cycle Graph

Let $\mathscr{C}_n$ denote the cycle graph with n vertices, where $n \geq 2$. An *independent set* of the graph $\mathscr{C}_n$ is a subset of $\mathscr{C}_n$ containing no pair of adjacent vertices. We associate each independent set of $\mathscr{C}_n$ with its *binary representation*, a cyclic binary string $v_1, v_2, \dots, v_n$ such that no two adjacent entries are both 1 (where "cyclic" means v_1 and v_n are considered adjacent).



$\longleftrightarrow$ 10100100

Toggle Groups

- Many actions of interest in dynamical algebraic combinatorics can be expressed as compositions of toggles, detailed in [Str18]. Let $\mathscr{L}$ be a collection of "allowed" subsets of a set E. For each $k \in E$, the *toggle at k* is the function $\tau_k : \mathscr{L} \to \mathscr{L}$ defined as

$$\tau_k(E) = \begin{cases} E \cup \{k\} & \text{if } k \notin E \text{ and } E \cup \{k\} \in \mathscr{L} \\ E \setminus \{k\} & \text{if } k \in E \text{ and } E \setminus \{k\} \in \mathscr{L} \\ E & \text{otherwise.} \end{cases}$$

- In this work, our set $\mathscr{L}$ of allowed subsets is the set of independent sets of $\mathscr{C}_n$, with vertex set $E = [n] = \{1, 2, \dots, n\}$. The *toggle group* is the group generated by $\{\tau_1, \tau_2, \dots, \tau_n\}$.
- Over the years, we have observed interesting properties for toggle actions on order ideals of various posets and independent sets of various graphs. Toggling independent sets of a path graph is analyzed in [JR18], making the similar action on the cycle graph natural to study.
- Our action $\tau : \mathscr{L} \to \mathscr{L}$ applies the toggles left-to-right (in the binary representation of the independent set) $\tau := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$.

$$001010 \xrightarrow{\tau_1} 101010 \xrightarrow{\tau_2} 101010 \xrightarrow{\tau_3} 100010 \xrightarrow{\tau_4} 100010 \xrightarrow{\tau_5} 100000 \xrightarrow{\tau_6} 100000$$

τ

An Example Orbit and the Original Conjecture

- Given an initial string $x^{(0)}$, let $x^{(1)} = \tau(x^{(0)})$, $x^{(2)} = \tau(x^{(1)})$, and so on. Eventually, after some number m steps, we will return to our original string. That is, $x^{(m+i)} = x^{(i)}$ for all i.
- In the example on the right, $n = 12$ and $m = 15$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
$x^{(0)}$	1	0	1	0	1	0	0	0	1	0	1	0
$x^{(1)}$	0	0	0	0	0	1	0	0	0	0	0	1
$x^{(2)}$	0	1	0	1	0	0	1	0	1	0	0	0
$x^{(3)}$	0	0	0	0	1	0	0	0	0	1	0	1
$x^{(4)}$	0	1	0	0	0	1	0	1	0	0	0	0
$x^{(5)}$	0	0	1	0	0	0	0	0	1	0	1	0
$x^{(6)}$	1	0	0	1	0	1	0	0	0	0	0	0
$x^{(7)}$	0	1	0	0	0	0	1	0	1	0	1	0
$x^{(8)}$	0	0	1	0	1	0	0	0	0	0	0	1
$x^{(9)}$	0	0	0	0	0	1	0	1	0	1	0	0
$x^{(10)}$	1	0	1	0	0	0	0	0	0	0	1	0
$x^{(11)}$	0	0	0	1	0	1	0	1	0	0	0	1
$x^{(12)}$	0	1	0	0	0	0	0	0	1	0	0	0
$x^{(13)}$	0	0	1	0	1	0	1	0	0	1	0	1
$x^{(14)}$	0	0	0	0	0	0	0	1	0	0	0	0
Sum:	**3**	**4**	**5**	**3**	**4**	**5**	**3**	**4**	**5**	**3**	**4**	**5**

Theorem and Original Conjecture (DJMM)

- In any orbit, the period of the sum vector is odd.
- Given an odd $r > 1$, there exists an orbit with sum vector period r if and only if $r \mid n$ and $n \geq 4r$.

Orbits by Period of Sum Vector

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
number of orbits	1	1	1	2	2	3	3	5	5	10	18	17	19	35	37	64	94	133	379	433	333	590	848	1355
sum vector period 1	1	1	1	2	2	3	3	5	5	10	9	17	19	29	37	64	73	133	114	211	333	590	701	1240
sum vector period 3	0	0	0	0	0	0	0	0	0	0	9	0	0	6	0	0	21	0	0	222	0	0	147	0
sum vector period 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	265	0	0	0	0	115

Scrolls and Ticker Tapes

- Given a starting independent set $x^{(0)}$, the *scroll* $\mathcal{S}$ is the vertically bi-infinite table where each row is an independent set $x^{(i)}$. The set of entries of $\mathcal{S}$ containing 1 (called *live entries*) is denoted $\mathsf{Live}(\mathcal{S})$.
- We use the coordinates (i,j) to refer to the row $x^{(i)}$ and column v_j. For notation convenience, when $n < i \leq 2n$, we say $(i,j) = (i-n, j+1)$.
- The *ticker tape* is the sequence of entries read left to right and top to bottom (like a book).
- The (i,j) entry of the scroll corresponds to the $(ni+j)$ entry of the ticket tape.
- The ticker tape of the orbit shown on the left below is

$$\dots, \underbrace{X_{-6}, X_{-5}, X_{-4}, X_{-3}, X_{-2}, X_{-1}, X_0}_{1,0,1,0,0,0,0}, \underbrace{X_1, X_2, X_3, X_4, X_5, X_6, X_7}_{1,0,1,0,0,0,0}, \underbrace{X_8, X_9, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}}_{1,0,1,0,0,0,0}, \dots.$$

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(1)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(2)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(3)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(4)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(5)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(6)}$	0	0	0	0	1	0	1	0	0	0	0
…	…	…	…	…	…	…	…	…	…	…	…

period of ticker tape: 7

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
$x^{(0)}$	1	0	1	0	1	0	0	0	1	0	1	0
$x^{(1)}$	0	0	0	0	0	1	0	0	0	0	0	1
$x^{(2)}$	0	1	0	1	0	0	1	0	1	0	0	0
$x^{(3)}$	0	0	0	0	1	0	0	0	0	1	0	1
$x^{(4)}$	0	1	0	0	0	1	0	1	0	0	0	0
$x^{(5)}$	0	0	1	0	0	0	0	0	1	0	1	0
$x^{(6)}$	1	0	0	1	0	1	0	0	0	0	0	0
$x^{(7)}$	0	1	0	0	0	0	1	0	1	0	1	0
$x^{(8)}$	0	0	1	0	1	0	0	0	0	0	0	1
$x^{(9)}$	0	0	0	0	0	1	0	1	0	1	0	0
$x^{(10)}$	1	0	1	0	0	0	0	0	0	0	1	0
$x^{(11)}$	0	0	0	1	0	1	0	1	0	0	0	1
$x^{(12)}$	0	1	0	0	0	0	0	0	1	0	0	0
$x^{(13)}$	0	0	1	0	1	0	1	0	0	1	0	1
$x^{(14)}$	0	0	0	0	0	0	0	1	0	0	0	0
…	…	…	…	…	…	…	…	…	…	…	…	…

period of ticker tape: 45

Theorem (DJMM)

Let $\mathcal{S}$ be a scroll. The period of the ticker tape is

$$\frac{\mathsf{Scale}(\mathcal{S})}{\deg(\mathcal{S})\,\mathrm{codeg}(\mathcal{S})}$$

(definitions given later).

Snakes

For live entry (i,j), another live entry is either:

- in position $(i, j+2)$ (called a 2 step), or
- in position $(i+1, j+1)$ (called a D step).

The *successor* function $s : \mathsf{Live}(\mathcal{S}) \to \mathsf{Live}(\mathcal{S})$ sends (i,j) to the unique element of

$$\{(i,j+2), (i+1,j+1)\} \cap \mathsf{Live}(\mathcal{S}).$$

The orbits of the action $\langle s \rangle$ on $\mathsf{Live}(\mathcal{S})$ are called *snakes*.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(1)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(2)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(3)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(4)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(5)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(6)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(7)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(8)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(9)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(10)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(11)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(12)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(13)}$	0	0	0	0	1	0	1	0	0	0	0

2 snakes slither: $(2D)^3$

Co-snakes

For live entry (i,j), another live entry is either:

- in position $(i+2, j-1)$ (called an S step), or
- in position $(i+2, j-2)$ (called an L step).

The *co-successor* function $c : \mathsf{Live}(\mathcal{S}) \to \mathsf{Live}(\mathcal{S})$ sends (i,j) to the unique element of

$$\{(i+2,j-2), (i+2,j-1)\} \cap \mathsf{Live}(\mathcal{S}).$$

The orbits of the action $\langle c \rangle$ on $\mathsf{Live}(\mathcal{S})$ are called *co-snakes*.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(1)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(2)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(3)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(4)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(5)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(6)}$	0	0	0	0	1	0	1	0	0	0	0
$x^{(7)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(8)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(9)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(10)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(11)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(12)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(13)}$	0	0	0	0	1	0	1	0	0	0	0

6 co-snakes co-slither: S^2

Slithers and Co-Slithers

- Consider a live entry (i,j). The *slither* is the sequence of steps 2 and D following the successor function of (i,j) until one reaches a position on the same co-snake as (i,j).
- Consider a live entry (i,j). The *co-slither* is the sequence of steps S and L following the co-successor function of (i,j) until one reaches a position on the same snake as (i,j).
- Slithers and co-slithers are equivalence classes up to cyclic shift, so $(2D)^3$ can also be written $(D2)^3$.
- The exponent on the slither (resp. co-slither) is called the *degree* $\deg(\mathcal{S})$ (resp. *co-degree* $\mathrm{codeg}(\mathcal{S})$) of the scroll $\mathcal{S}$. It is the number of times the smallest periodic string is repeated to form the slither (resp. co-slither). In the example, $\deg(\mathcal{S}) = 3$ and $\mathrm{codeg}(\mathcal{S}) = 2$.
- The *scale* of a scroll, written $\mathsf{Scale}(\mathcal{S})$, is the minimal (ticker tape) distance between live entries on the same snake and the same co-snake.

Proposition (DJMM)

All snakes have the same slither. All co-snakes have the same co-slither.

Theorem (DJMM)

The slither of any scroll has an odd number of D's.

Theorem (DJMM)

- The set $\mathsf{Live}(\mathcal{S})$ is a torsor for the *snake group*, which has presentation

$$\langle s, c \mid sc = cs,\ s^\beta = c^\alpha \rangle$$

where $\mathcal{S}$ has α snakes and β co-snakes. That is, the snake group acts freely and transitively on $\mathsf{Live}(\mathcal{S})$.
- Furthermore, for any $i \in \mathsf{Live}(\mathcal{S})$,

$$s^\beta(i) - i = c^\alpha(i) - i = \mathsf{Scale}(\mathcal{S}).$$

Orbit Tables and Ouroboroi

- An *orbit table* is a partial scroll where toggling maps the bottom string to the top string.
- The image of a snake (resp. co-snake) when allowed to wrap from top to bottom is called an *ouroboros* (resp. *co-ouroboros*)
- The name was inspired by the ancient symbol of a snake swallowing its tail (drawing from 1478 alchemy text drawing by Theodoros Pelecanos, image taken from Wikipedia).
- Below to the right, the two snakes form one ouroboros and the six co-snakes form two co-ouroboroi.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
$x^{(0)}$	1	0	1	0	0	0	0	1	0	1	0
$x^{(1)}$	0	0	0	1	0	1	0	0	0	0	1
$x^{(2)}$	0	1	0	0	0	0	1	0	1	0	0
$x^{(3)}$	0	0	1	0	1	0	0	0	0	1	0
$x^{(4)}$	1	0	0	0	0	1	0	1	0	0	0
$x^{(5)}$	0	1	0	1	0	0	0	0	1	0	1
$x^{(6)}$	0	0	0	0	1	0	1	0	0	0	0

Determining all Scrolls/ Ticker Tapes/ Orbit Tables for a Given n

Theorem (DJMM)

For a fixed n, we can construct all scrolls/ orbit tables/ ticker tapes that begin with a live entry through the following procedure:

1. Take a solution to the equation:

$$2\beta_T + 3\alpha_S + 4\alpha_L = n + 1$$

with $\beta_T, \alpha_S, \alpha_L \geq 0$ and $\alpha_S + \alpha_L > 0$.

2. Choose any sequence of $\beta_D = 2(\alpha_S + \alpha_L) - 1$ instances of D and β_T instances of 2. This gives the slither of each snake.
3. Choose any sequence of α_S instances of S and α_L instances of L. This gives the co-slither of each co-snake.

β_T	α_S	α_L	β_D	Slither	Co-slither
5	0	1	1	*22222D*	*L*
3	0	2	3	*222DDD*	*LL*
3	0	2	3	*22D2DD*	*LL*
3	0	2	3	*22DD2D*	*LL*
3	0	2	3	*2D2D2D*	*LL*
1	0	3	5	*2DDDDD*	*LLL*
4	2	0	3	*2222DDD*	*SS*
4	2	0	3	*222D2DD*	*SS*
4	2	0	3	*222DD2D*	*SS*
4	2	0	3	*22D22DD*	*SS*
4	2	0	3	*22D2D2D*	*SS*
2	2	1	5	*22DDDDD*	*SSL*
2	2	1	5	*2D2DDDD*	*SSL*
2	2	1	5	*2DD2DDD*	*SSL*
0	2	2	7	*DDDDDDD*	*SSLL*
0	2	2	7	*DDDDDDD*	*SLSL*

References

[DJMM] C. Defant, M. Joseph, M. Macauley, and A. McDonough. Torsors from toggling independent sets. *In preparation.*

[JR18] M. Joseph and T. Roby. Toggling independent sets of a path graph. *Electron J. Combin.*, 25(1)1–18, 2018.

[Str18] J. Striker. Rowmotion and generalized toggle groups. *Discrete Mathematics & Theoretical Computer Science*, 20, 2018.