FPSAC2022 poster: Stable sets in flag spheres

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Abstract

We provide lower and upper bounds on the minimum size of a maximum stable set over graphs of flag spheres, as a function of the dimension of the sphere and the number of vertices.
Further, we use stable sets to obtain an improved Lower Bound Theorem for the face numbers of flag spheres.

Main invariant: $\alpha(d,n)$

For a graph $G=(V,E)$, $S\subseteq V$ is stable (a.k.a. independent) if the induced graph $G[S]$ has no edges.
The maximal size of a stable set in G is denoted $\alpha(G)$.
The clique complex is denoted $cl(G)$ and its geometric realization $||cl(G)||$.

$$\alpha(d,n) := \min(\alpha(G):\ |V(G)|=n,\ ||cl(G)||\cong S^{d-1})$$

Question: For fixed d, what is the growth of $\alpha(d,n)$ as $n\to\infty$?

$\alpha(d,n)$=?

Conjecture: For every $d\geq 2$ and $n\geq 2d$,

$$\alpha(d,n)=\lceil\frac{n+d-3}{2(d-1)}\rceil.$$

Case $d=2$: True, easy.
Case $d=3$: True, lower bound via the 4-Color-Theorem (4CT), upper bound via construction, $cl(W_{3,k})$, see Fig.1.
Case $d=4$: Upper bound holds via construction.

Theorem: Let $d\geq 4$ and $n\geq 2d$. Then

$$\frac{1}{4}n^{\frac{1}{d-2}}\leq\alpha(d,n)\leq\lceil\frac{\lceil\frac{n}{\lfloor d/4\rfloor}\rceil+1}{6}\rceil.$$

Lower bound: Ramsey type inductive argument, uses the 4CT for the base case $d=4$.
Upper bound: take joins of the construction of flag 3-spheres $cl(W'_{4,k})$, see Fig.2, and up to 3 suspensions, to reach a flag sphere of dimension $d-1$.

The graphs $W_{d,k}$

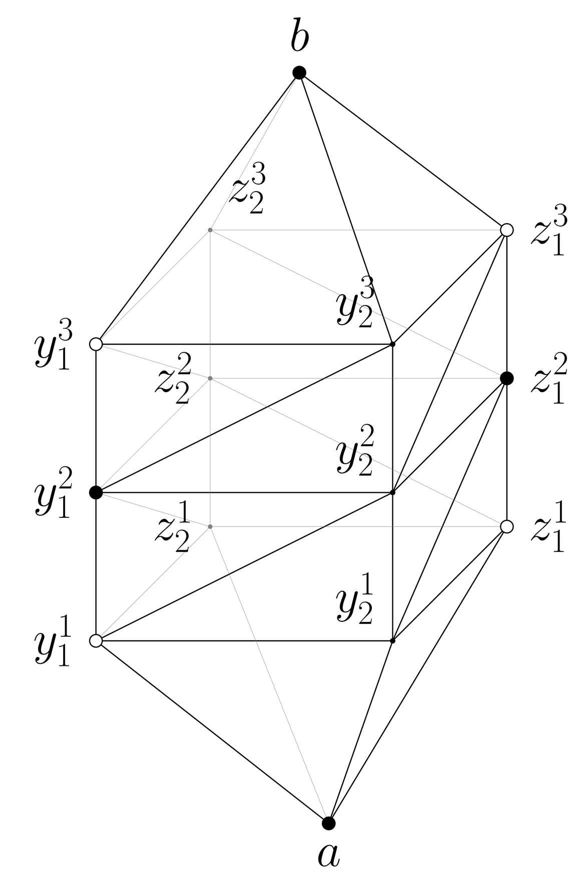

Figure 1: The graph $W_{3,3}$ is depicted. The bold black and bold white vertices indicate stable sets of size $\alpha(W_{3,3})=4$. The shaded edges indicate edges that are not visible from a front view of the depicted realization of the flag 2-sphere $cl(W_{3,3})$ in 3-space.

Fix an integer $d\geq 2$. For $k\geq 1$ let $G=W_{d,k}$ be the following graph:
Vertices: $V(W_{d,k})=\{a,b\}\cup X_1\cup\ldots\cup X_k$
where the sets $X_1,\ldots,X_k,\{a,b\}$ are pairwise disjoint and $|X_i|=2d-2$ for every $i\in\{1,\ldots,k\}$.
Denote $X_i=\{y_1^i,\ldots,y_{d-1}^i,z_1^{i'},\ldots,z_{d-1}^i\}$.
Edges:

- a is complete to X_1 and b is complete to X_k and there are no other edges incident with a,b.
- For every i, the induced graph $W_{d,k}[X_i]$ is the 1-skeleton of the $(d-1)$-dimensional crosspolytope, a.k.a. the graph of the octahedral $(d-2)$-sphere, with non-edges $y_1^iz_1^i,\ldots,y_{d-1}^iz_{d-1}^i$.
- X_i is anticomplete to X_j if $|i-j|>1$.
- For $i\in\{1,\ldots,k-1\}$ and $s,t\in\{1,\ldots,d-1\}$ let us say that the pair $(y_s^iz_s^i,y_t^{i+1}z_t^{i+1})$ is positive if $y_s^iy_t^{i+1}$ and $z_s^iz_t^{i+1}$ are edges, and $y_s^iz_t^{i+1}$ and $z_s^iy_t^{i+1}$ are non-edges, and negative if $y_s^iy_t^{i+1}$ and $z_s^iz_t^{i+1}$ are non-edges, and $y_s^iz_t^{i+1}$ and $z_s^iy_t^{i+1}$ are edges. Then the pair $(y_s^iz_s^i,y_t^{i+1}z_t^{i+1})$ is positive if $t\geq s$ and negative if $t<s$.
- All pairs of vertices of $W_{d,k}$ that are not mentioned above are non-edges.

Observation: For all $k\geq 1$,

$$||cl(W_{3,k})||\cong S^2,$$

and

$$\alpha(W_{3,k})=\lceil\frac{|V(W_{3,k})|}{4}\rceil.$$

From $W_{4,k}$ to $W'_{4,k}$

$W_{4,k}$ induces a cell structure on the 3-sphere, consisting of tetrahedra with a vertex a or b and of triangular prisms consisting of a triangle on X_i and the corresponding triangle on X_{i+1} (the corresponding vertices differ only in the superscript).
All these triangular prisms are triangulated by considering all tertrahedra defined by cliques of $W_{4,k}$ on this set of 6 vertices, except for the following two (for a fixed $1\leq i\leq k-1$):
$y_1^i,z_2^i,y_3^i;y_1^{i+1},z_2^{i+1},y_3^{i+1}$ and its "antipodal prism" $z_1^i,y_2^i,z_3^i;z_1^{i+1},y_2^{i+1},z_3^{i+1}$. We add the edge $y_1^iz_2^{i+1}$ to triangulate the first, and the edge $z_1^iy_2^{i+1}$ to triangulate the second (such added edge is "bent' inside the prism, see Fig.2).
Denote the resulted graph by $W'_{4,k}$.

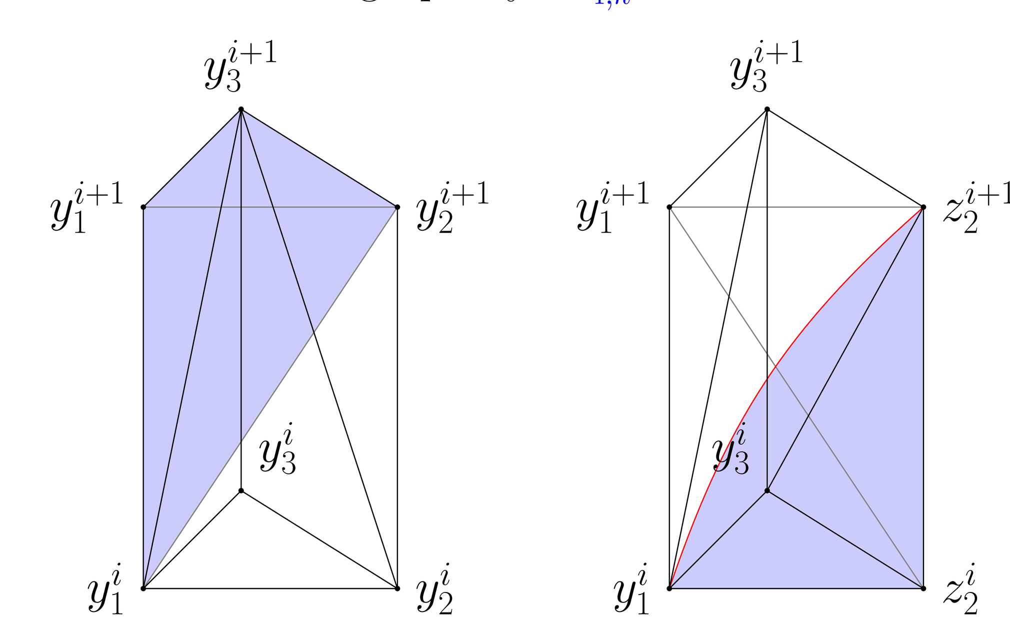

Figure 2: Two triangular prisms with the induced graphs on their vertices. The grey edges indicate edges not visible from a front view of the depicted realization embeded in 3-space. The red edge is bent inside the right prism. In purple are sample induced tetrahedra. Note that in each prism, its clique complex triangulates it.

Observation: for all $k\geq 1$,

$$||cl(W'_{4,k})||\cong S^3,$$

and

$$\alpha(W'_{4,k})=\lceil\frac{|V(W'_{4,k})|+1}{6}\rceil.$$

The Lower Bound Theorem: flag case

Barnette's LBT,'71: For all $d\geq 3$, all $1\leq i\leq d-1$, and every simplicial $(d-1)$-sphere Δ on n vertices,

$$f_i(\Delta)\geq f_i(S(d,n)),$$

where $S(d,n)$ is a **stacked** $(d-1)$-sphere on n vertices.
Reduction: $f_1(\Delta)\geq dn-\binom{d+1}{2}$.
So, asymptotically: Fix d. For every $\epsilon>0$, if n is large enough then $f_1(\Delta)\geq(d-\epsilon)n$.
Gal's conjecture,'05: For all $d\geq 3$, and every flag $(d-1)$-sphere Δ on n vertices,

$$f_1(\Delta)\geq(2d-3)n-2d(d-2).$$

Theorem:
For all $d\geq 6$, and n large enough, each n-vertex flag $(d-1)$-sphere Δ has at least $(d+\frac{0.987}{2d+1})n$ edges.

Proof sketch: via graph rigidity

- We will choose the largest $\epsilon=\epsilon(d)>0$ for which $f_1<(d+\epsilon)n$ yields a contradiction.
- By Turán's theorem, for $\Delta=cl(G)$, G has an independent set I of size $|I|\geq\frac{n}{2(d+\epsilon)+1}$.
- Assume $d\geq 5$. By Kalai's proof of the LBT, $g_2:=f_1-dn+\binom{d+1}{2}$ is the dimension of the **stress space** of a generic geometric embedding of G in $\mathbb{R}^d$.
- Further, the closed star of each vertex v contains a stress such that some edge containing v has a nonzero weight.
- Picking one such stress per vertex in I gives an independent set of stresses. Thus,
$$\epsilon n+\binom{d+1}{2}\geq f_1-dn+\binom{d+1}{2}\geq|I|\geq\frac{n}{2(d+\epsilon)+1}.$$
- Solve the quadric for ϵ.

Conjecture: For all $d\geq 5$, the graph of every flag $(d-1)$-sphere is $(d+1)$-rigid.

If true, then $f_1\geq(d+1)f_0-\binom{d+2}{2}$ would follow, for flag spheres of dimension $d-1\geq 4$.
Reduction: via the standard Cone and Gluing lemmas in Graph Rigidity, the case $d=5$ suffices.