Set Partitions, Tableaux, and Subspace Profiles of Regular Diagonal Operators

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0. Notation

λ : Integer partition of n.

$\Pi_n(\lambda)$: Set partitions of $[n]$ of shape λ.

$\mathrm{Tab}(\lambda)$: Standard tableaux of shape λ.

$b_\lambda(q)$: Polynomials indexed by integer partitions.

$S(n,m)$: Stirling numbers of the second kind.

$S_q(n,m)$: q-Stirling numbers of the second kind.

$c(T)$: Statistic on standard tableaux.

1. Counting set partitions

$$|\Pi_n(\lambda)| = \sum_{T\in \mathrm{Tab}(\lambda)} c(T).$$

$$S(n,m) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=m}} \sum_{T\in \mathrm{Tab}(\lambda)} c(T).$$

$$B_n = \sum_{T\in \mathrm{Tab}_n} c(T).$$

Bell number

2. Specializations of $b_\lambda(q)$

$b_\lambda(q)$:

- $q=1$ → #set partitions of shape λ.
- $q=0$ → #standard tableaux of shape λ.
- $q=-1$ → #shifted standard tableaux of shape λ.

3. Subspace profiles

Δ : Regular diagonal operator on $\mathbf{F}_q^n$.

W : Subspace of $\mathbf{F}_q^n$.

W has Δ-profile $\mu = (\mu_1, \mu_2, \dots)$ if

$$\dim(W + \Delta W + \cdots + \Delta^{j-1}W) = \mu_1 + \cdots + \mu_j \ (\forall j \geq 1)$$

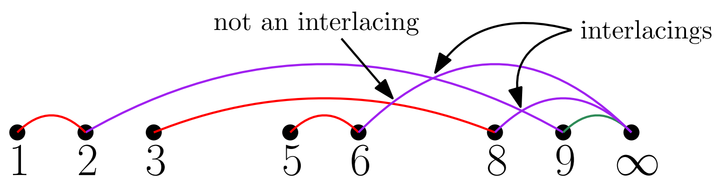
$\sigma(\mu)$: #subspaces with Δ-profile μ.

$$\sigma(\mu) = \binom{n}{|\mu|}(q-1)^{\sum_{j\geq 2}\mu_j} q^{\sum_{j\geq 2}\binom{\mu_j}{2}} b_{\mu'}(q)$$

4. $b_\lambda(q)$ via a statistic on set partitions

An interlacing of a set partition is a crossing of j-th arcs for some j.

Example: $\mathcal{A} = 129|38|56$. Interlacing number $v(\mathcal{A}) = 2$.



$$b_\lambda(q) = \sum_{\mathcal{A}\in \Pi_n(\lambda)} q^{v(\mathcal{A})}$$

5. q-Stirling numbers

$$S_q(n,m) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=m}} q^{\sum_i (i-1)(\lambda_i-1)} b_\lambda(q)$$

6. λ has parts ≤ 2

$b_\lambda(q) \leftrightarrow$ Catalan triangle associated to q-Hermite orthogonal polynomials.

$$b_{(2^m)}(q) = T_m(q)$$

$T_m(q)$: Generating polynomial for chord diagrams on $2m$ points by number of crossings.