Weak faces and a weight-formula for highest weight modules, via parabolic-PSP for roots

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Abstract

- Let $\mathfrak{g}$ be a general complex Kac–Moody Lie algebra, with root system Δ and simple roots Π. The partial sum property (PSP) for Δ says: every positive root is an ordered sum of simple roots, with each partial sum also a root.
- We show a parabolic generalization of the PSP, which we call as the parabolic-PSP: Suppose we have $\emptyset \neq S \subseteq \Pi$ and a positive root β involving some simple roots from S in its expansion. Then: *β is an ordered sum of roots, each involving exactly one simple root from S and such that each partial sum of the ordered sum is also a root.*
- We show three applications of the parabolic-PSP to weights of highest weight $\mathfrak{g}$-modules V:
  (1) A minimal description for weights of all simple V.
  (2) A uniform formula for weights of arbitrary V.
  (3) Determining and showing the equivalence of weak faces and $(\{2\};\{1,2\})$-closed subsets of weight-sets of all V.

Notations for Kac–Moody algebras

All the vector spaces are over complex numbers $\mathbb{C}$. Throughout, $\mathfrak{g}$ stands for a general Kac–Moody Lie algebra over $\mathbb{C}$. We fix for $\mathfrak{g}$:

- Cartan subalgebra $\mathfrak{h}$ (with dual $\mathfrak{h}^*$), root system $\Delta = \Delta^+ \sqcup \Delta^- \subset \mathfrak{h}^*$.
- Simple roots $\Pi = \{\alpha_i \mid i \in \mathcal{I}\} \subset \Delta$ and co-roots $\Pi^\vee = \{\alpha_i^\vee \mid i \in \mathcal{I}\} \subset \mathfrak{h}$, Chevalley generators: e_i (raising), f_i (lowering), $\alpha_i^\vee\ \forall\, i \in \mathcal{I}$, and Weyl group $W = \langle s_i \mid i \in \mathcal{I}\rangle$ generated by the simple reflections s_i.
- $\mathcal{I} = \{$nodes in Dynkin diagram of $\mathfrak{g}\} = \{$indices of simple roots$\}$.
  Example: when $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ we set $\mathcal{I} = \{1,2\}$, and when $\mathfrak{g}$ is affine, say $\widehat{\mathfrak{sl}_3}(\mathbb{C})$, we set $\mathcal{I} = \{0,1,2\}$.
- Triangular decomposition $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, and root space decomposition $\mathfrak{h} \oplus \bigoplus_{\beta\in\Delta}\mathfrak{g}_\beta$, for root spaces $\mathfrak{g}_\beta := \{x \in \mathfrak{g} \mid h\cdot x = \beta(h)x\ \forall\, h \in \mathfrak{h}\}$.

We say $\mathfrak{g}$ is of finite type, if it is a finite-dimensional simple Lie algebra.

Parabolic partial sum property

Begin by recalling the well-known fundamental property of root systems Δ:

Partial sum property

Every root in Δ is an ordered sum of simple roots, such that all the partial sums are also roots.

We show a novel parabolic-generalization of the PSP, with applications to weights of highest weight $\mathfrak{g}$-modules.

Two examples in type E_6 demonstrating the parabolic-PSP

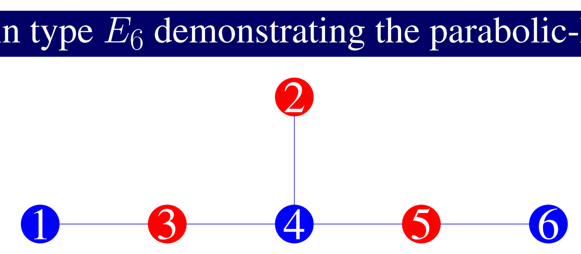

Figure 1: Dynkin diagram of $\mathfrak{g}$ of type E_6, with $\mathcal{I} = \{1,\dots,6\}$ and $I = \{2,3,5\}$ in red colour.

Example 1: $\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6 = (\alpha_1+\alpha_3+\alpha_4)+(\alpha_5+\alpha_6)$.

Example 2: $\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$
$= (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)+(\alpha_4+\alpha_5+\alpha_6)$
$= (\alpha_1+\alpha_2)+(\alpha_3)+(\alpha_4+\alpha_5)+(\alpha_4+\alpha_5+\alpha_6)$.

To state the parabolic-PSP for $\emptyset \neq I \subseteq \mathcal{I}$ we need:

- A generalized height function– $\operatorname{ht}_I(\sum_{i\in\mathcal{I}} c_i\alpha_i) := \sum_{i\in I} c_i$ for $c_i \in \mathbb{C}$.
- The set of unit-I-heights roots– $\Delta_{I,1} := \{\beta \in \Delta \mid \operatorname{ht}_I(\beta) = 1\} \subseteq \Delta^+$.

Theorem 1: Parabolic partial sum property (Teja, [1], 2020)

Let Δ be a Kac–Moody root system, and fix $\emptyset \neq I \subseteq \mathcal{I}$. Suppose β is a positive root with $m = \operatorname{ht}_I(\beta) > 0$.

1. Then there exist roots $\gamma_1,\dots,\gamma_m \in \Delta_{I,1}$ such that
$$\beta = \sum_{j=1}^m \gamma_j \quad \text{and} \quad \sum_{j=1}^i \gamma_j \in \Delta^+ \quad \text{for all } 1 \le i \le m.$$
In other words, every root with positive I-height is an ordered sum of roots, each with unit I-height, and with each partial sum also a root.
2. In fact, the root space $\mathfrak{g}_\beta$ is spanned by the right normed Lie words
$$[e_{\gamma_m},[\cdots,[e_{\gamma_2},e_{\gamma_1}]\cdots]] \text{ for } \gamma_t \in \Delta_{I,1},\ e_{\gamma_t} \in \mathfrak{g}_{\gamma_t}\ \forall t, \text{ and } \sum_{t=1}^m \gamma_t = \beta.$$

Three Applications of the parabolic-PSP

For this, we need to define parabolic Verma modules $M(\lambda, J)$.

Notations for highest weight modules

- Fix $\lambda \in \mathfrak{h}^*$. The integrable directions for λ are $J_\lambda := \{j \in \mathcal{I} \mid \lambda(\alpha_j^\vee) \in \mathbb{Z}_{\geq 0}\}$.
- $M(\lambda)$ and $L(\lambda)$ are the Verma module and the simple highest weight module over $\mathfrak{g}$ with highest weight λ.
- $M(\lambda) \twoheadrightarrow V$ denotes a nonzero highest weight $\mathfrak{g}$-module V with highest weight λ.
- Fix $M(\lambda) \twoheadrightarrow V$. For $\mu \in \mathfrak{h}^*$ the μ-weight space of V, and the weight-set of V:
$$V_\mu := \{v \in V \mid h\cdot v = \mu(h)v\ \forall\, h \in \mathfrak{h}\}, \quad \text{and} \quad \operatorname{wt}V := \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}.$$
The convex hull over (reals) $\mathbb{R}$ of $\operatorname{wt}V$ is denoted $\operatorname{conv}(\operatorname{wt}V)$.
- The parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to $J \subseteq \mathcal{I}$ is $\mathfrak{p}_J := \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}_J^-$, with $\mathfrak{n}_J^-$ the subalgebra of $\mathfrak{n}^-$ generated by $f_j\ \forall\, j \in J$ and negative roots Δ_J^-.
- For $J \subseteq J_\lambda$, the maximum integrable highest weight $\mathfrak{p}_J$-module with highest weight λ is denoted $L_J^{\max}(\lambda)$.
- The key objects in the weight-formulas below for simples, are the parabolic Verma modules $M(\lambda, J)$ for $J \subseteq J_\lambda$:
$$M(\lambda,J) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_J)} L_J^{\max}(\lambda) \simeq \frac{M(\lambda)}{\sum_{j\in J} U(\mathfrak{g}) f_j^{\lambda(\alpha_j^\vee)+1} M(\lambda)_\lambda}. \tag{1}$$
These are universal among J-integrable modules – *Weyl character formula*:
$$\operatorname{char} M(\lambda,J) = \sum_{w\in W_J} \frac{(-1)^{\ell(w)} e^{w\bullet\lambda}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}, \quad \forall \lambda \in \mathfrak{h}^*,\ J \subseteq J_\lambda.$$

Appln. 1: Minimal description for $\operatorname{wt}L(\lambda)$

The definition of parabolic Verma modules implies the weight-formula:

Minkowski decomposition: $\operatorname{wt}M(\lambda,J) = \operatorname{wt}L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+)$. (2)

Here, given subsets C, D of a vector space, their Minkowski sum/difference is $C \pm D := \{c \pm d \mid c \in C, d \in D\}$. If $C = \{\lambda\}$, we write $C \pm D = \lambda \pm D$.

- Weights of integrable $L(\lambda)$ (λ dominant & integral: $J_\lambda = \mathcal{I}$) were well-known.
- Weights of non-integrable $L(\lambda)$ are recently computed:

Theorem 2 (Khare [J. Alg., 2016] and Dhillon–Khare [J. Alg., 2022])

For all $\lambda \in \mathfrak{h}^*$, $\operatorname{wt}L(\lambda) = \operatorname{wt}M(\lambda,J_\lambda) = \operatorname{wt}L_{J_\lambda}^{\max}(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{J_\lambda}^+)$. (3)

The parabolic-PSP yields minimal generators for the $\mathbb{Z}_{\geq 0}$-cones above, and thereby:

Theorem 3: A minimal description for weights of highest weight simples (Teja, [1], 2020)

$$\Delta^+ \setminus \Delta_J^+ \subset \mathbb{Z}_{\geq 0}\Delta_{J^c,1}\ \forall\, J \subseteq \mathcal{I}. \quad \text{So, } \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+) = \mathbb{Z}_{\geq 0}\Delta_{J^c,1}. \tag{4}$$
$$\text{Hence,} \quad \operatorname{wt}M(\lambda,J) = \operatorname{wt}L_J^{\max}(\lambda) - \mathbb{Z}_{\geq 0}\Delta_{J^c,1}. \tag{5}$$
$$\text{In particular,} \quad \operatorname{wt}L(\lambda) = \operatorname{wt}L_{J_\lambda}^{\max}(\lambda) - \mathbb{Z}_{\geq 0}\Delta_{J_\lambda^c,1}. \tag{6}$$

This is novel even in finite type.

Appln. 2: Weight-formula for all $\operatorname{wt}V$

This extends (6) from $L(\lambda)$ to *all highest weight modules:*

Theorem 4: A weight-formula for all highest weight modules (Teja, [1], 2020)

$$\operatorname{wt}V = \left[\operatorname{wt}V \cap (\lambda - \mathbb{Z}_{\geq 0}\Pi_{J_\lambda})\right] - \mathbb{Z}_{\geq 0}\Delta_{J_\lambda^c,1}, \quad \text{for all } M(\lambda) \twoheadrightarrow V. \tag{7}$$

This reduces the problem of determining weights for arbitrary $M(\lambda) \twoheadrightarrow V$, to finding those with λ dominant and integral.

This too is novel in finite type – e.g. even for $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$.

Weak faces & $(\{2\};\{1,2\})$-closed sets

Fix subsets $\emptyset \neq Y \subseteq X$ of a real vector space, henceforth.

- Recall for X convex: Y is a face of X if given vectors $y_1,\dots,y_n \in Y$ and $x_1,\dots,x_m \in X$, and scalars $r_i, t_j \in \mathbb{R}_{\geq 0}$, the following holds
$$\sum_{i=1}^n r_i y_i = \sum_{j=1}^m t_j x_j \text{ and } \sum_{i=1}^n r_i = \sum_{j=1}^m t_j > 0 \implies x_j \in Y\ \forall t_j \neq 0.$$
(For polyhedra X, same as exposed faces – maximizers of linear functionals.)
- Now let $0 \neq \mathbb{A} \subseteq (\mathbb{R},+)$ be an additive subgroup, and X be arbitrary. We define a weak face of X to be Y as above, with $\mathbb{R}_{\geq 0}$ replaced by $\mathbb{A}_{\geq 0} := \mathbb{A} \cap \mathbb{R}_{\geq 0}$ – *for any* $\mathbb{A}$.

Upshot: Weak faces are discrete combinatorial analogues of faces.

- We say Y is a $(\{2\};\{1,2\})$-closed subset of X if
$$(y_1)+(y_2) = (x_1)+(x_2) \text{ for } y_1, y_2 \in Y,\ x_1, x_2 \in X \implies x_1, x_2 \in Y.$$

Combinatorial interpretation for $(\{2\};\{1,2\})$-closed subsets:

- Say X is the set of lattice points in a lattice polytope, and $Y \subseteq X$ is a subset of "infected" lattice points, such that if $y \in Y$ is the average of two points in X, then the two points catch the "infection" from y.
- More precisely, if two pairs of points have the same average, and one pair is colored, then the color spreads to the other pair.
- We aim at understanding the extent to which the spread happens.

Origins of weak faces and $(\{2\};\{1,2\})$-closed subsets: Introduced by Chari and co-authors in 2000s – applications in representation theory:

- Constructing Koszul algebras.
- Constructing nilpotent ideals in parabolic subalgebras of $\mathfrak{g}$.
- Obtaining character formulas of Kirillov–Reshetikhin modules over untwisted quantum affine algebras $U_q(\widehat{\mathfrak{g}})$ at the specialization $q = 1$.

Appln. 3: Weak faces, $(\{2\};\{1,2\})$-closed subsets of roots and weights

The first part here easily follows from the definitions:

Proposition 5 (Khare, [J. Alg., 2016])

Suppose $\emptyset \neq Y \subseteq X$ in a real vector space.

1. Each of the following implies the next:
   (i) Y is an exposed face of X – i.e., maximizes a linear functional.
   (ii) Y is a face of X – i.e., a weak $\mathbb{R}$-face.
   (iii) Y is a weak face of X, for some $\mathbb{A} \subseteq (\mathbb{R},+)$.
   (iv) Y is $(\{2\};\{1,2\})$-closed in X.
   However, (ii) does not imply (i) even for convex $X \subset \mathbb{R}^2$.
2. Say $\mathfrak{g}$ is of finite type, and $M(\lambda) \twoheadrightarrow V$ is any simple highest weight module or parabolic Verma module. Setting $X = \operatorname{wt}V$, the subsets satisfying (i)–(iii) are equivalent, and are also precisely:
   (v) $\quad w\left[(\lambda - \mathbb{Z}_{\geq 0}\Pi_I) \cap \operatorname{wt}V\right] \quad$ for all $w \in W_{I_V}$, $I \subsetneq \mathcal{I}$. (8)

Here, I_V is the integrability of V, i.e. $I_V = \{j \in J_\lambda \mid f_j^{\lambda(\alpha_j^\vee)+1} V_\lambda = 0\}$.

Questions:

- (When) Is (iv) also equivalent to (i)–(iii), (v)?
- What happens for other highest weight modules?
- What if $\mathfrak{g}$ is of infinite (affine, Kac–Moody) type?

Now proved in complete generality:

Theorem 6: Weak faces & $(\{2\};\{1,2\})$-closed subsets of weights (Teja, [2], 2021)

For any Kac–Moody $\mathfrak{g}$, any $\lambda \in \mathfrak{h}^*$, and any module $M(\lambda) \twoheadrightarrow V$:

1. For $X = \operatorname{wt}V$, the five classes of subsets of $\operatorname{wt}V$ in (i)–(v) above, are equivalent.
2. For $X = \operatorname{conv}(\operatorname{wt}V)$, the classes of subsets of $\operatorname{wt}V$ in (i)–(iv) above, are equivalent – and they are equivalent to:
   (v′) the convex hulls of the subsets in (8) in (v).
3. Similar equivalences hold for root systems: $X = \Delta, \Delta \sqcup \{0\}$.

References

[1] G.V. Krishna Teja, *Moving between weights of weight modules,* Preprint, arXiv.2012.07775, 2020.

[2] G.V. Krishna Teja, *Weak faces of highest weight modules and root systems*, Preprint, arXiv:2106.14929, 2021.