Multiplication theorems for self-conjugate partitions

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Introduction

- A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of n is a positive integer sequence such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ and $|\lambda| := \sum_{i=1}^{\ell} \lambda_i = n$. $\mathcal{P} :=$ {integer partition}.
- Conjugate $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{\lambda_1})$ of a partition λ: partition whose Ferrers diagram is obtained by the reflection of the Ferrers diagram of λ along the main diagonal.
- Self-conjugate partition $\lambda \in \mathcal{SC}$: partition such that $\lambda = \lambda'$.
- A t-core is a partition with no hook length divisible by t, where the hook length is the number $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ for a box $(i,j) \in \lambda$, $\mathcal{H}(\lambda) :=$ {hook length}. For $t \in \mathbb{N}^*$, $\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod t\}$ is the set of all hook lengths divisible by t.

(a) Durfee square (b) Hook lengths (c) BG-rank (d) ε_u

Han–Ji multiplication Theorem (2011)

Let t be a positive integer and let ρ_1 be a function defined on $\mathbb{N}$. Let f_t be the following formal power series:

$$f_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th)$$

Then we have

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) = t \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty} \left(f_t(xq^t)\right)^t$$

Nekrasov–Okounkov formulas

Nekrasov–Okounkov formula (2006), Westbury (2006), Han (2008)
For any fixed complex number z, we have:

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = (q;q)_\infty^{z-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

Its modular analogue [Han–Ji (2011)]

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{z}{h^2}\right) = \frac{(q^t;q^t)_\infty^t}{(xq^t;xq^t)_\infty^{t-z/t} (q;q)_\infty}$$

Goals

- Provide a $\mathcal{SC}$ version of Han and Ji
- Discussions upon the parity of t
- Modular analogue of the $\mathcal{SC}$ Nekrasov–Okounkov formula

Main Theorem

Set t an even integer and let $\tilde{\rho}_1$ be a function defined on $\mathbb{Z} \times \{-1, 1\}$. Set also $f_t(q)$ the formal power series defined by:

$$f_t(q) := \sum_{\nu \in \mathcal{P}} q^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \tilde{\rho}_1(th, 1)\tilde{\rho}_1(th, -1)$$

Then we have

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_u, \varepsilon_u)$$

$$= f_t(x^2q^{2t})^{t/2} (q^{2t};q^{2t})_\infty^{t/2} (-bq;q^4)_\infty (-q^3/b;q^4)_\infty$$

A Multiplication Theorem for t odd

Let t be a positive odd integer and set $\mathrm{BG}_t := \{\lambda \in \mathcal{SC} \mid \forall i \in \{1, \dots, d\}, t \nmid h_{(i,i)}\}$. Set $\tilde{\rho}_1$ a function defined on $\mathbb{Z} \times \{-1, 1\}$. Let f_t be the formal power series defined in the main Theorem. Then we have

$$\sum_{\lambda \in \mathrm{BG}_t} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_u, \varepsilon_u) = f_t(x^2q^{2t})^{(t-1)/2} \frac{(q^{2t};q^{2t})_\infty^{(t-1)/2} (-q;q^2)_\infty}{(-q^t;q^{2t})_\infty}$$

A key tool: the Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega_t \in \mathcal{A} \mid \mathcal{H}_t(\omega_t) = \emptyset\}$
Littlewood decomposition: bijection such that

$$\lambda \in \mathcal{P} \mapsto (\omega_t, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

$$\mathcal{H}_t(\lambda) = t \bigcup_{i=0}^{t-1} \mathcal{H}(\nu^{(i)})$$

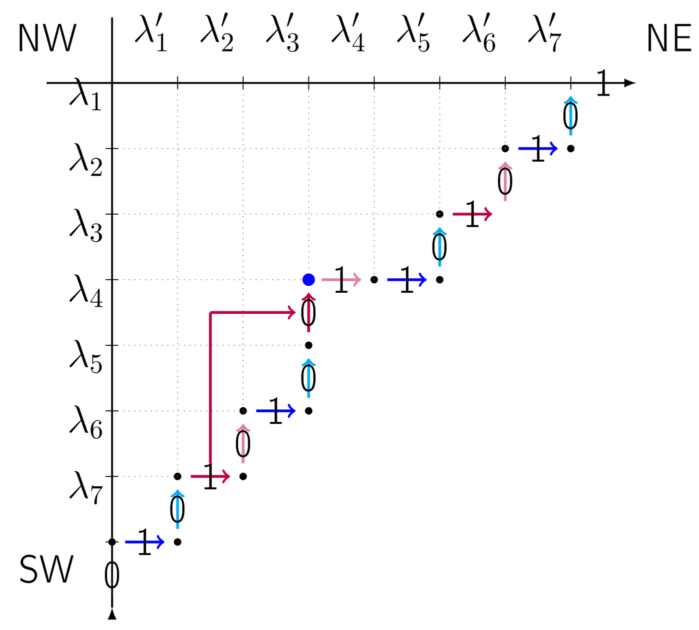
$$|\lambda| = |\omega_t| + t \sum_{i=0}^{t-1} |\nu^{(i)}|$$

Restriction to self-conjugate partitions:

(a) for t even: $\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{t/2}$

(b) for t odd:
$\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC}$

Remark: Cho–Huh–Sohn (2019)
$\lambda \in \mathcal{SC}^{(\mathrm{BG})} \mapsto \mu \in \mathcal{P}$ bijection such that
$|\lambda| = 4|\mu| + m(m+1)/2$



Corollary

Let t be a positive even integer and let ρ_1 be a function defined on $\mathbb{N}$. Let f_t be the formal power series defined as:

$$f_t(q) := \sum_{\nu \in \mathcal{P}} q^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \rho_1(th)^2$$

Then we have

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) = f_t(x^2q^{2t})^{t/2} (q^{2t};q^{2t})_\infty^{t/2} (-bq;q^4)_\infty (-q^3/b;q^4)_\infty$$

Modular $\mathcal{SC}$ Nekrasov–Okounkov

Let t be a positive even integer and set $\tilde{\rho}_1$ a function defined on $\mathbb{Z} \times \{-1, 1\}$. Let f_t be the formal power series defined in the main Theorem. Then we have

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \left(1 - \frac{z}{h^2}\right)^{1/2} = \sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \left(1 - \frac{z}{h_u \varepsilon_u}\right)$$

$$= (x^2q^{2t};x^2q^{2t})_\infty^{(z/t-t)/2} (q^{2t};q^{2t})_\infty^{t/2} (-bq;q^4)_\infty (-q^3/b;q^4)_\infty$$

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