The canonical complex of the weak order

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Joint work with Vincent Pilaud (CNRS & École Polytechnique)

Definition

A (finite) **lattice** L is a (finite) poset where every family X of elements of L has a **join** $\bigvee X$ (smallest upper bound) and a **meet** $\bigwedge X$ (greatest lower bound).

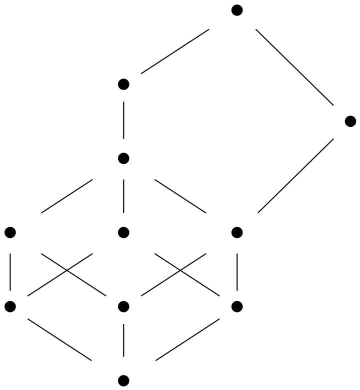
Definition

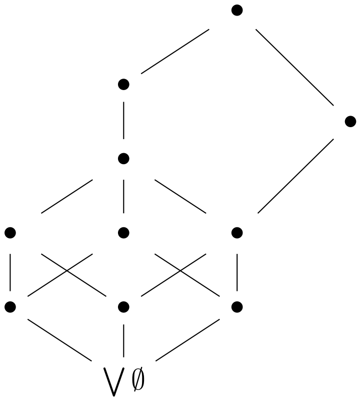
The **canonical join representation** of an element x is a subset $J \subseteq L$ such that:

- $\bigvee J = x$,
- $J' \subsetneq J \Rightarrow \bigvee J' \neq x$,
- J is *lowest* in L with these properties.

When it always exist, we call the lattice **join semidistributive**.

Irreducibility

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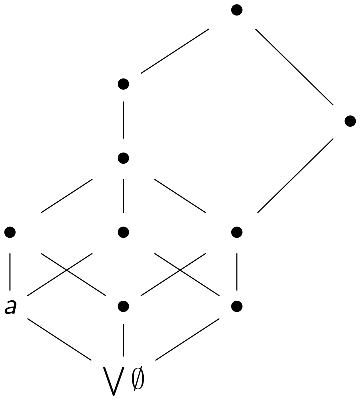
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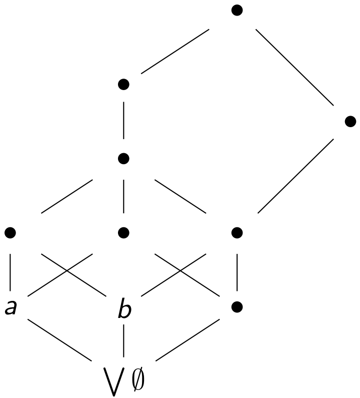
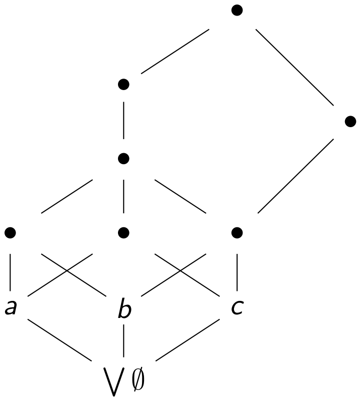
Irreducibility

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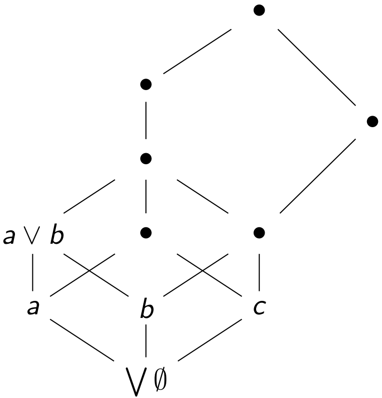
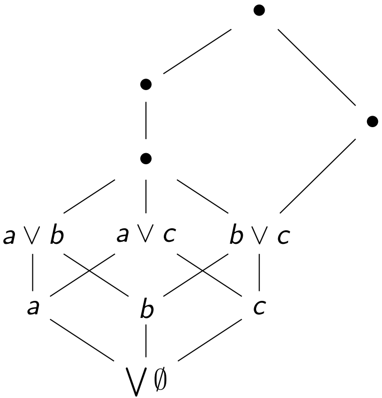
Irreducibility

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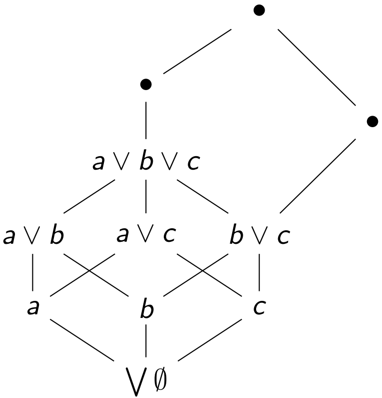
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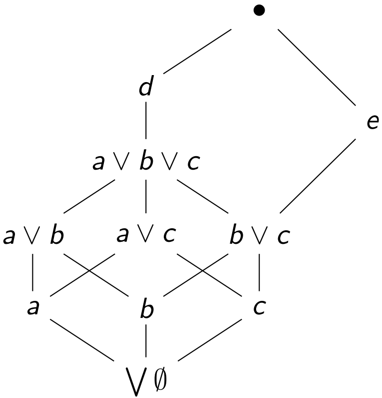
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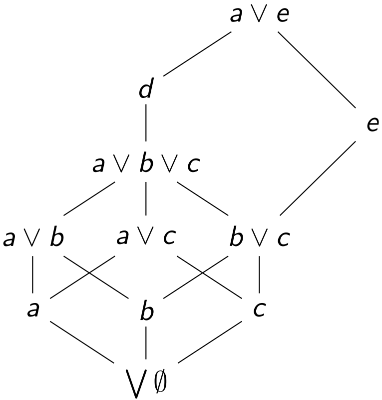
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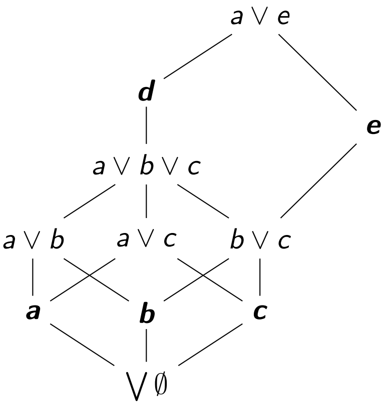

Definition

The elements that are their own canonical join representation are the **join irreducibles**. In finite lattices, they are those covering only one element. Canonical join representations are made of join irreducibles.

Definition (Reading '15, Barnard '19, '20)

The **canonical join complex** associated to a join semidistributive lattice L is the simplicial complex $\mathcal{CJC}(L)$ with:

- vertices $:=$ {join irreducibles},
- faces $:=$ {canonical join representations}.

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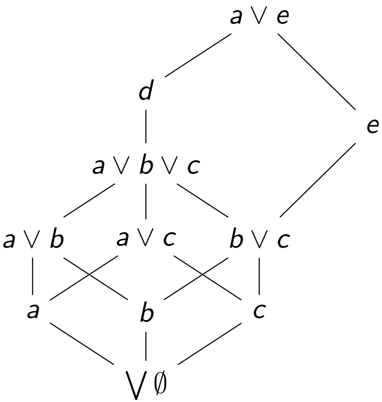
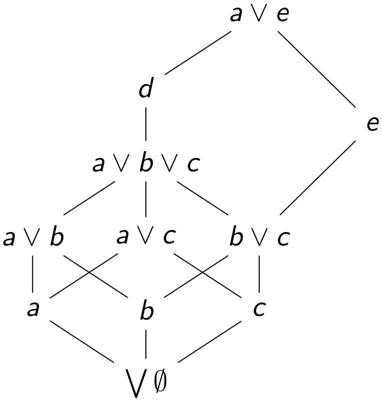
Theorem (Reading '15)

It is a **flag** *simplicial complex.*



Lattice L



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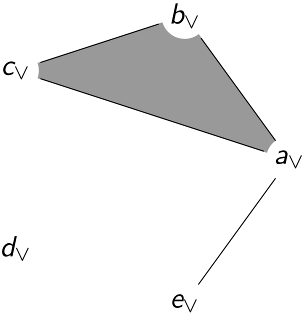
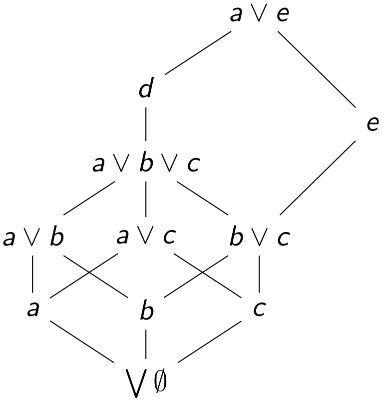


Canonical join complex $\mathcal{CJC}(L)$



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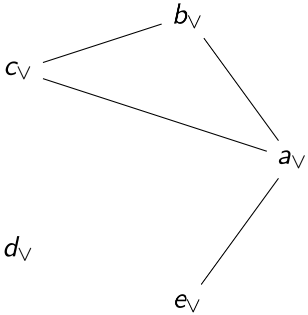

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Definition

A **lattice congruence** is an equivalence relation $\equiv$ on L such that $x \equiv x'$ and $y \equiv y'$ implies $x \vee y \equiv x' \vee y'$ and $x \wedge y \equiv x' \wedge y'$.

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Theorem (Reading '16)

A lattice congruence is characterized by the join irreducibles it contracts (merge with the one they cover). More precisely, there is a poset on join irreducibles called **forcing order** *such that all ideals of this poset correspond to a lattice congruence.*

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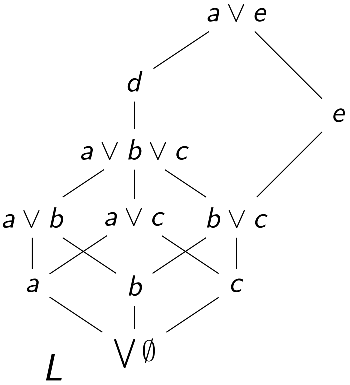
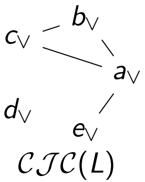
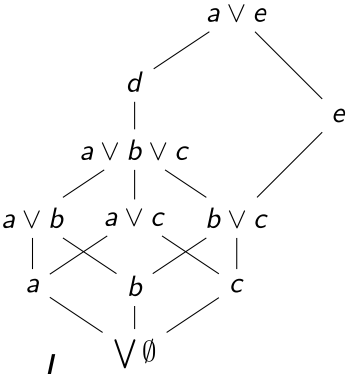
Theorem (Reading '15)

The canonical join complex behaves well with lattice congruences.

Lattice congruences







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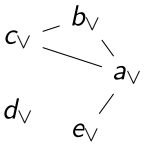
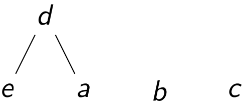
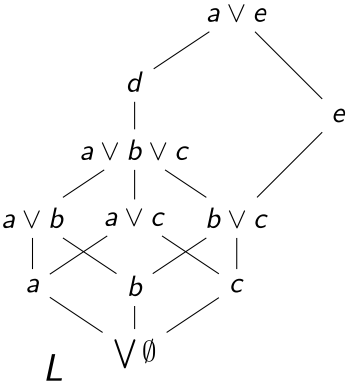


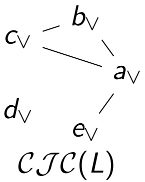
$\mathcal{CJC}(L)$

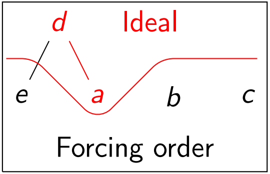


Forcing order

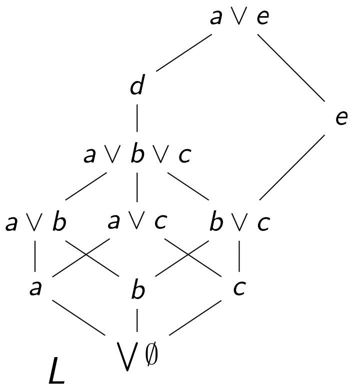
Lattice congruences

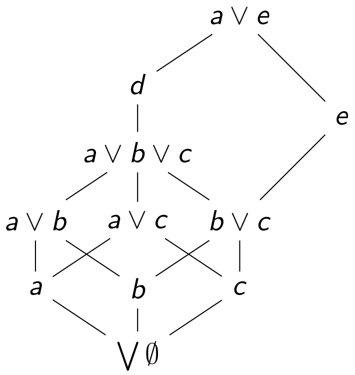


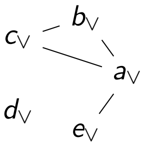




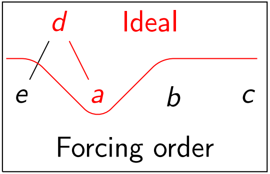
Lattice congruences

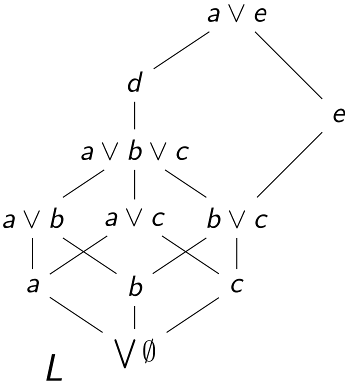






$\mathcal{CJC}(L)$

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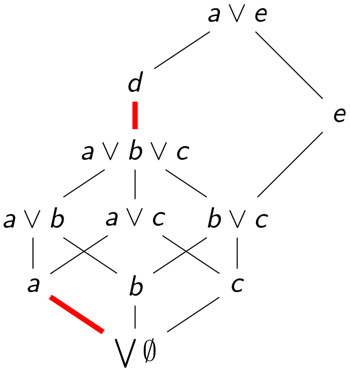
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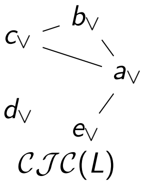
$b_\vee$
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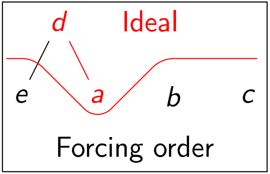
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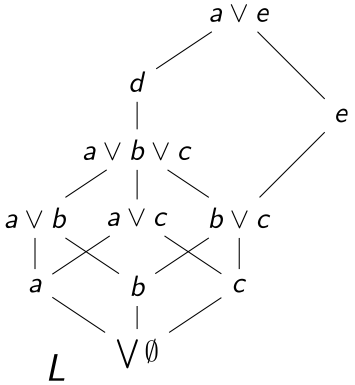
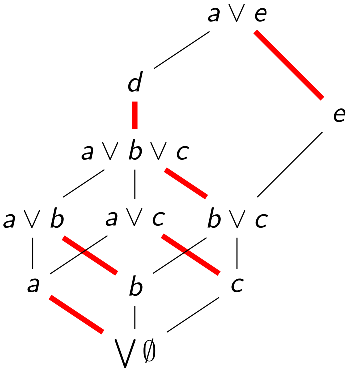
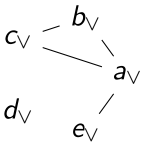
Forcing order







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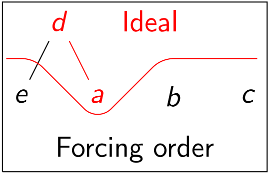

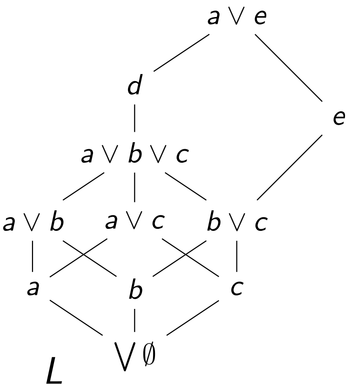
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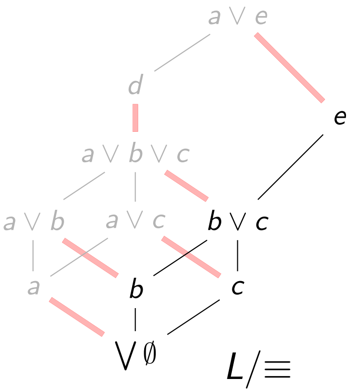
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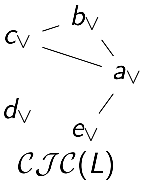
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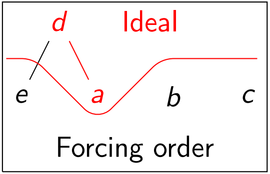
CJC(L)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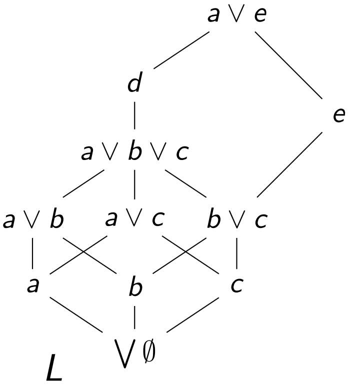
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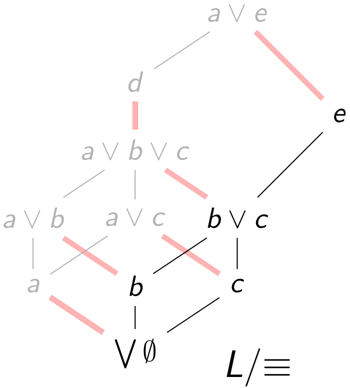
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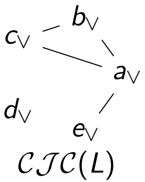
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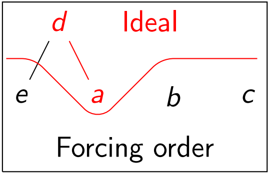
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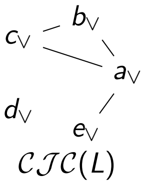
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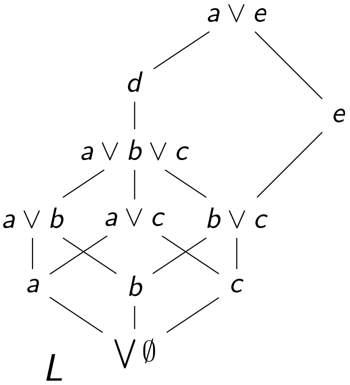
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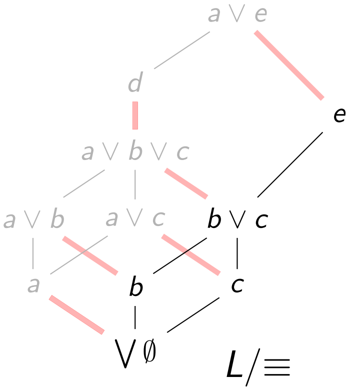
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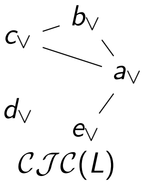
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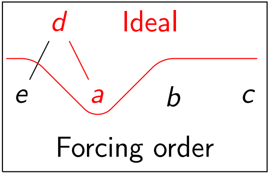
CJC(L)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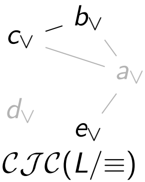
b∨
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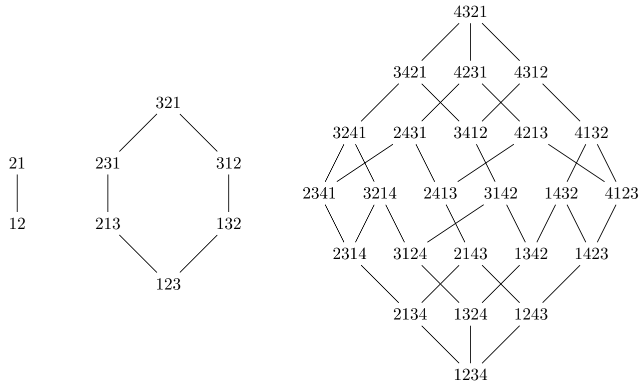
CJC(L/≡)

Proposition

The (right) weak order is a semidistributive lattice on permutations ordered by containment of their inversion sets.

$\mathrm{inv}(132) = \{(2,3)\} \subseteq \{(1,3),(2,3)\} = \mathrm{inv}(312)$

$132 \preccurlyeq 312$



A nice bijection

$\sigma = 526413$

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Permutation table:
$\{(\sigma_i, i) \mid i \in [n]\}$.

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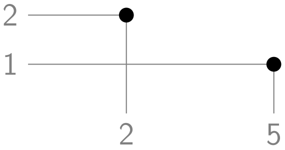

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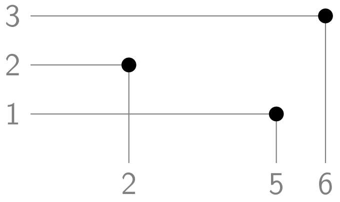

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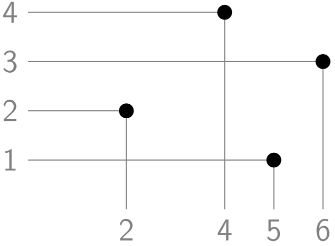

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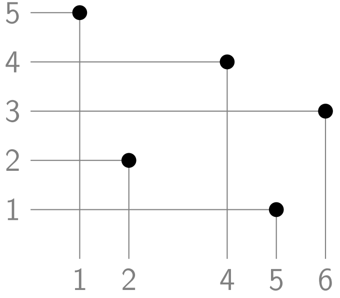

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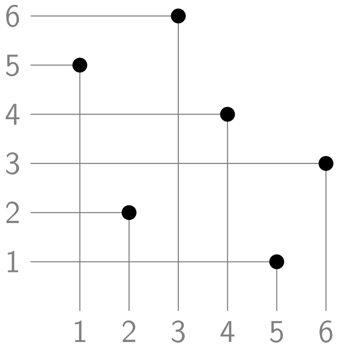

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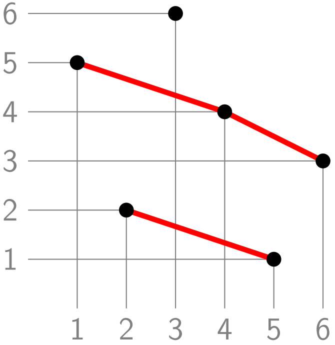

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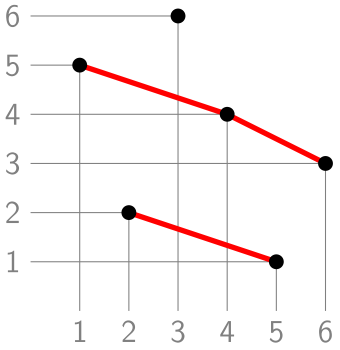

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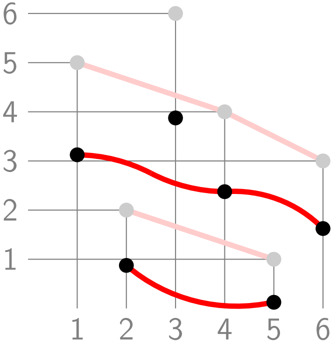

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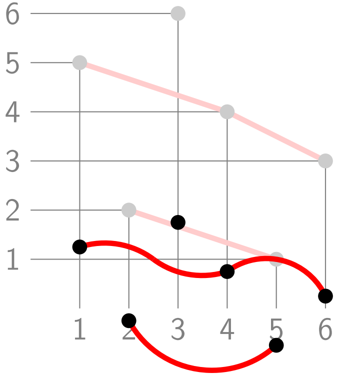

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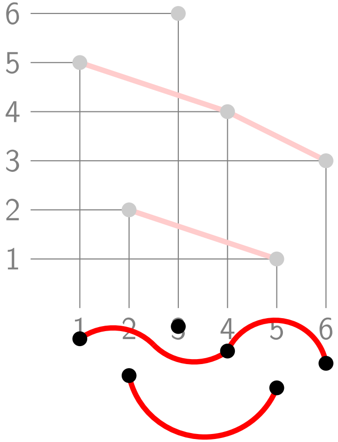

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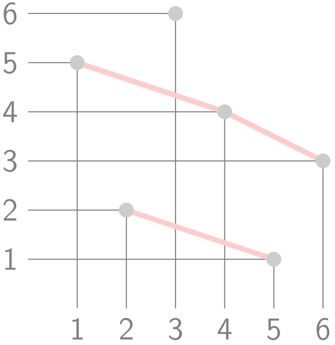

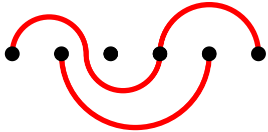

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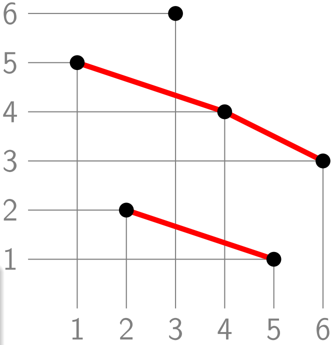

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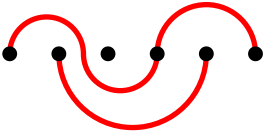

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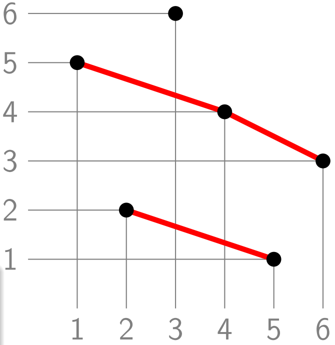

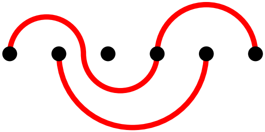

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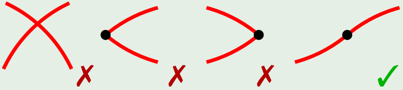

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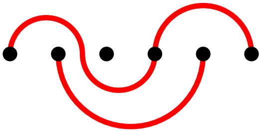

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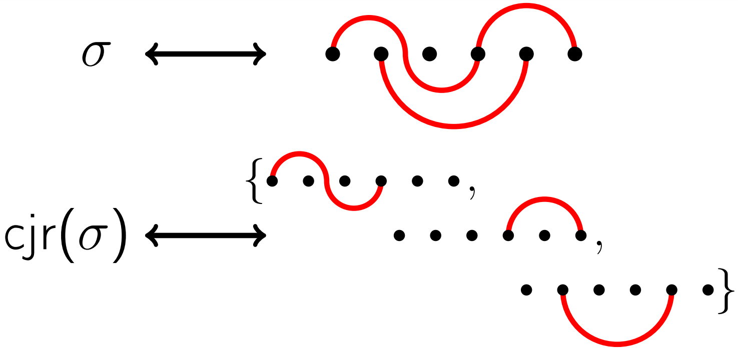

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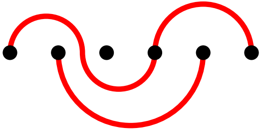
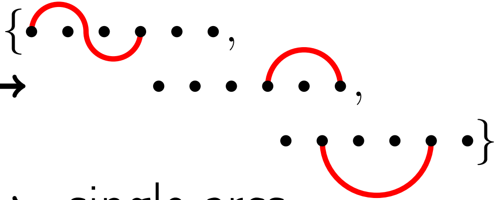
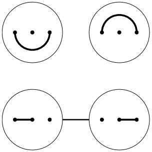
$\sigma \longleftrightarrow$ 

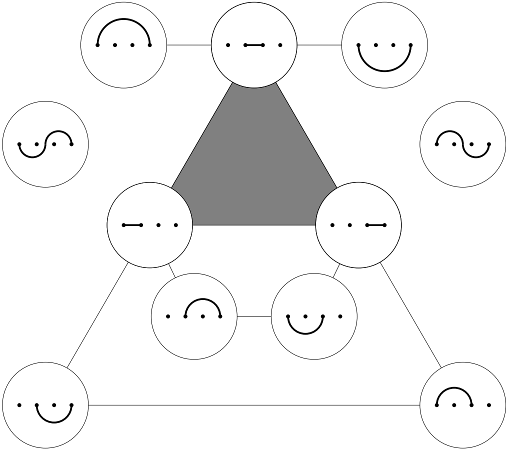
$\mathrm{cjr}(\sigma) \longleftrightarrow$ 

join irreducible permutations $\longleftrightarrow$ single arcs

Theorem (Reading '15)

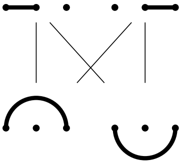
The canonical join complex of the weak order is isomorphic to the **non-crossing complex**.

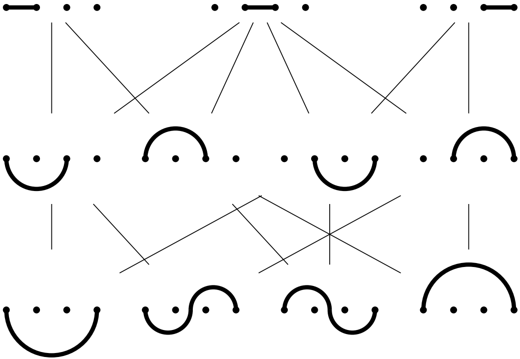




Proposition (Reading '15)

The forcing on arcs corresponds to the extension of arcs.





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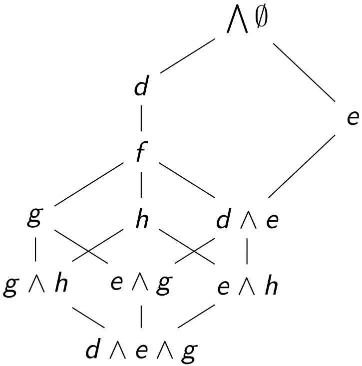

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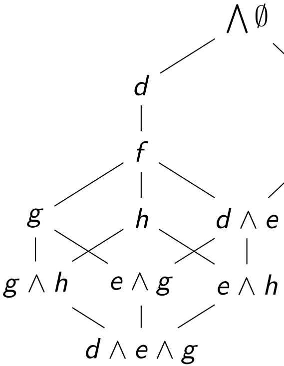

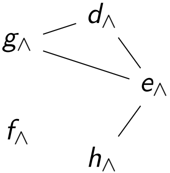

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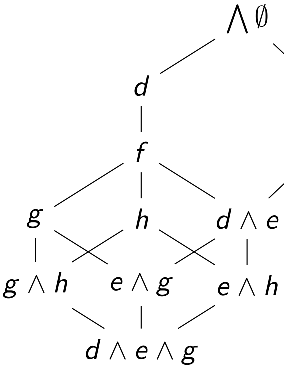
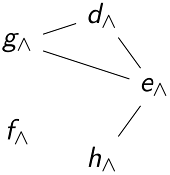
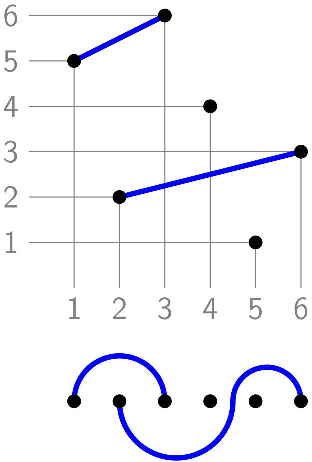






Definition (A., Pilaud '22+)

Canonical representation of an interval:

$$\mathrm{cr}([x,y]) := \mathrm{cjr}(x) \sqcup \mathrm{cmr}(y).$$

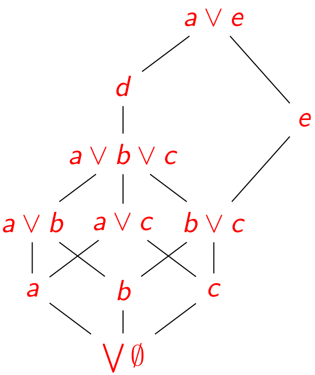
Canonical complex $\mathcal{CC}(L)$ of a semidistributive lattice L:

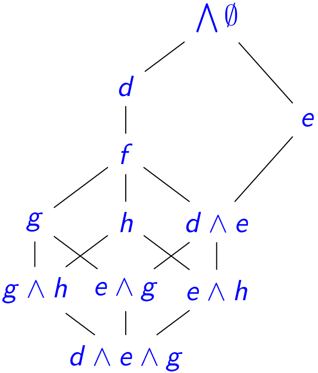
- vertices $:= \{\text{join irreducibles}\} \sqcup \{\text{meet irreducibles}\}$,
- faces $:= J \sqcup M$ such that:
  - J is a canonical join representation,
  - M is a canonical meet representation,
  - $\bigvee J \leq \bigwedge M$.

Theorem (A., Pilaud '22+)

The canonical complex is a well defined flag simplicial complex. It contains the canonical join and meet complexes. It behaves as well as those with respect to taking quotients of the lattice.

Back to our example





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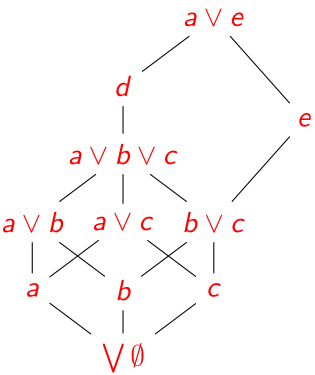

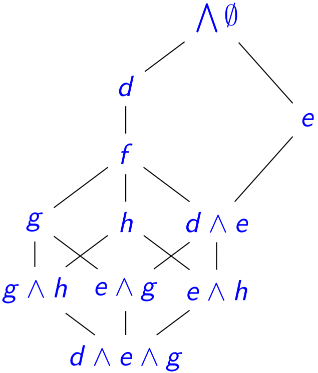

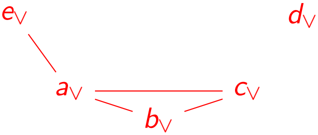

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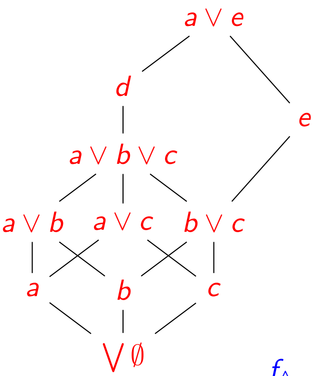

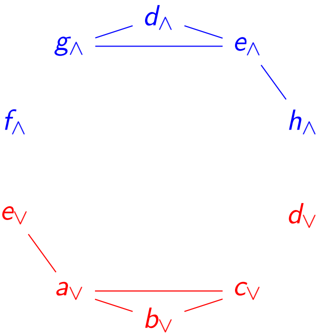

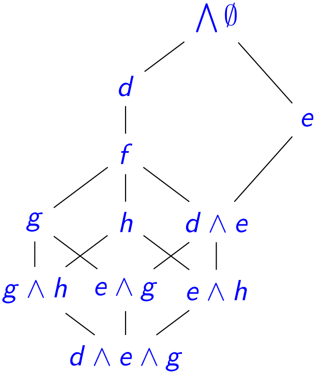

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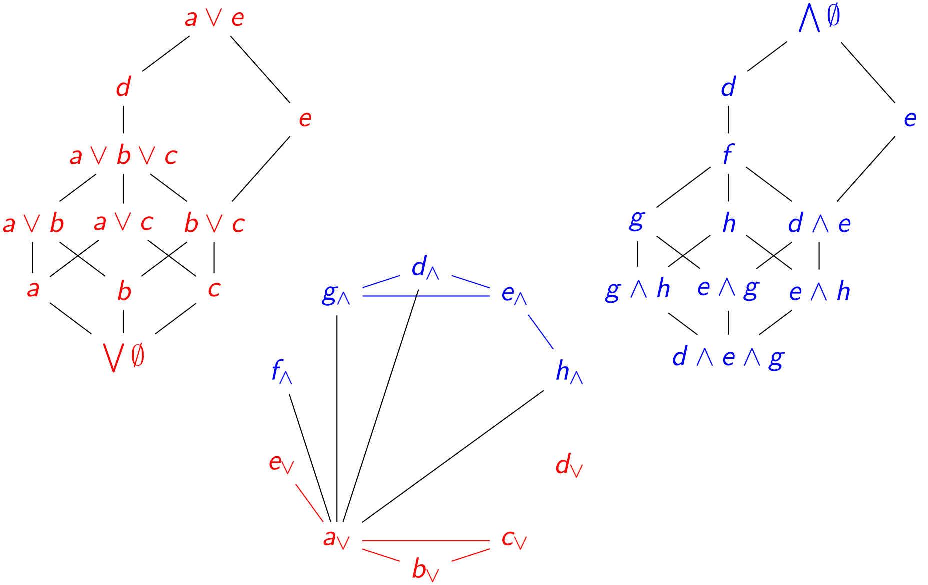

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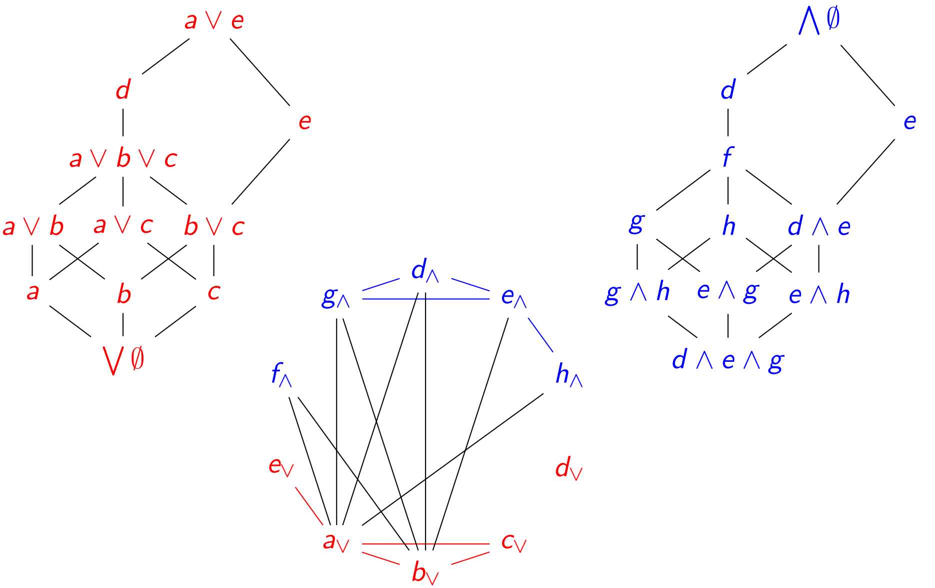

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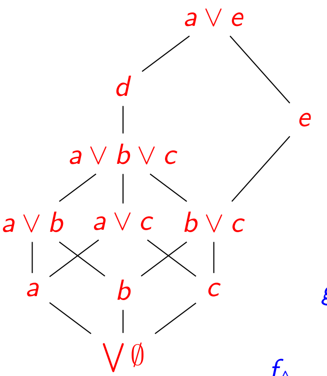

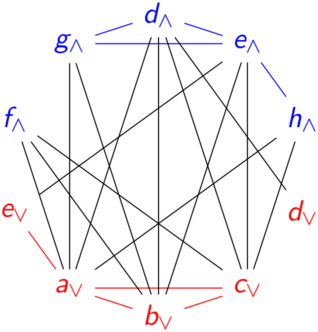

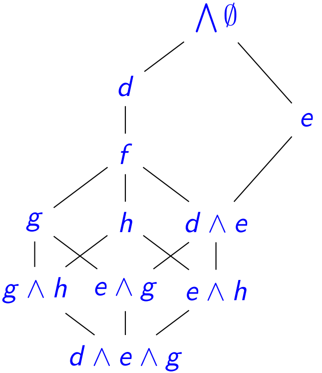

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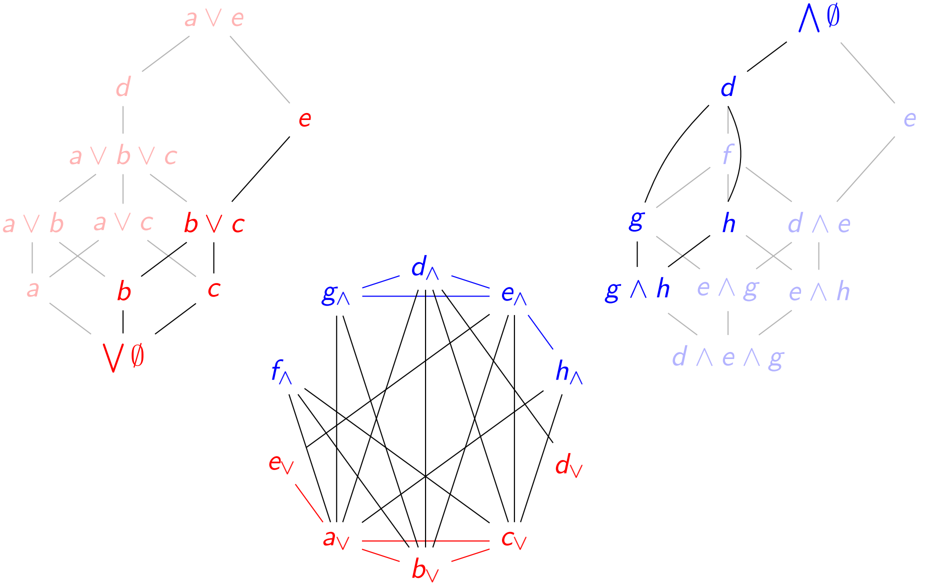

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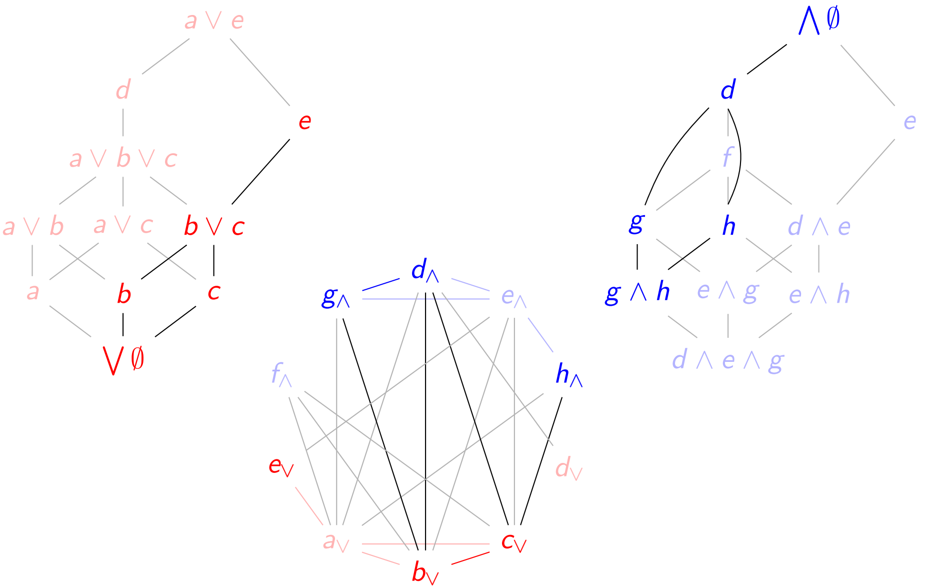
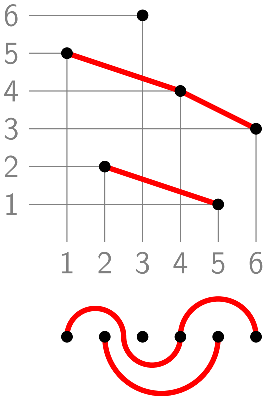
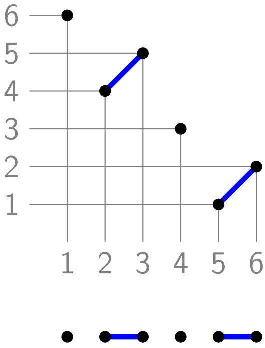


To the interval $[526413, 564231]$,





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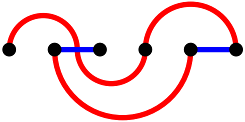

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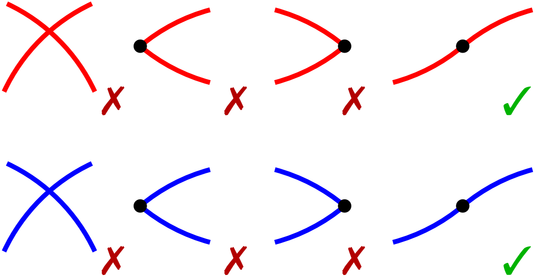

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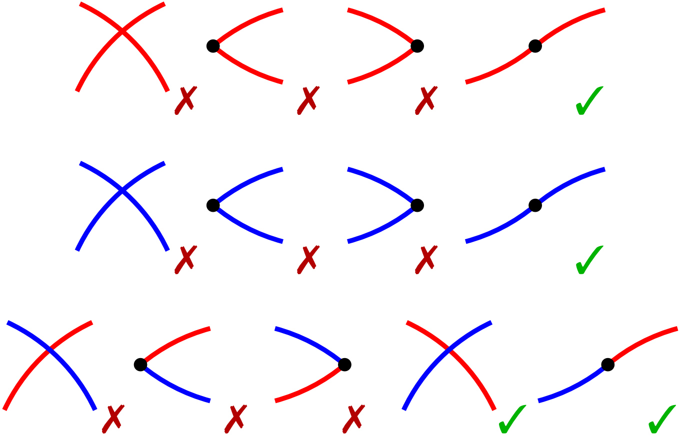

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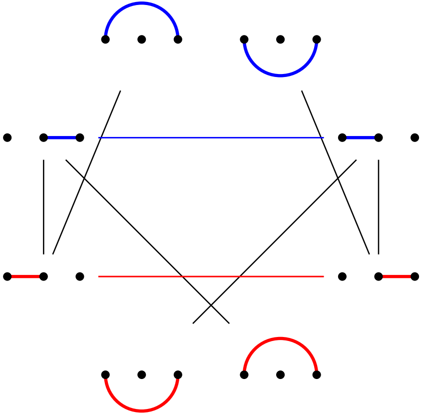
This bijection between intervals of the weak order and SCABs provides a combinatorial model for the canonical complex of the weak order: the **semi-crossing complex**.



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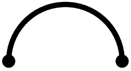
When $\equiv$ contracts all arcs but those shaped like , we recover the classical Kreweras complement on non-crossing partitions:

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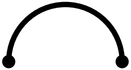
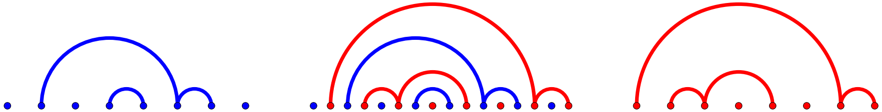
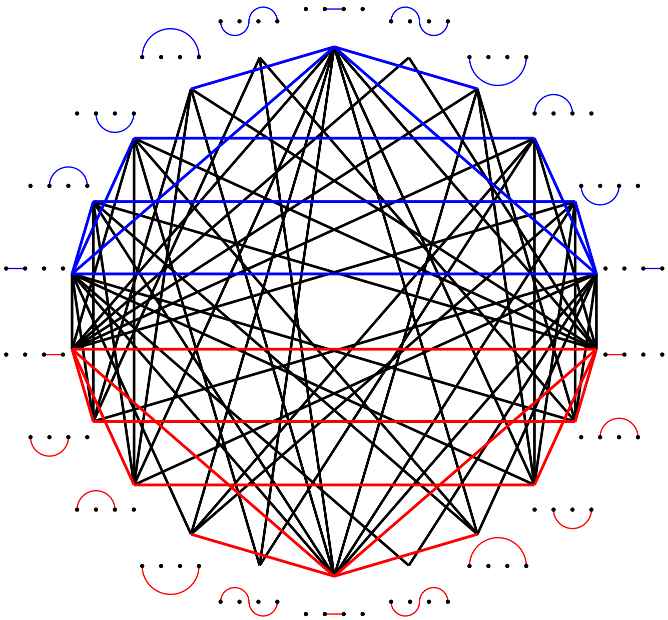
When $\equiv$ contracts all arcs but those shaped like , we recover the classical Kreweras complement on non-crossing partitions:





Thank you!