Biclosed sets in affine root systems

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What is a biclosed set?

Fix a real vector space V and a set of vectors $\Phi^+ \subseteq V$.

Definition (Dyer, Edgar, Papi)

$B \subseteq \Phi^+$ is **biclosed** if it satisfies the following two properties for all $\alpha, \beta, \gamma \in \Phi^+$ such that $\gamma = a\alpha + b\beta$ for some $a, b > 0$:

(Closed) If $\alpha \in B$ and $\beta \in B$, then $\gamma \in B$.

(Coclosed) If $\alpha \notin B$ and $\beta \notin B$, then $\gamma \notin B$.

Example: Biclosed sets in type A_2

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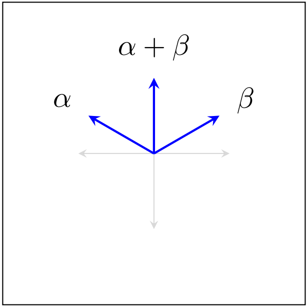

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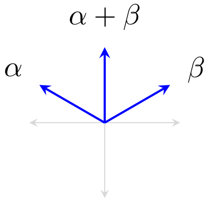

Φ^+ is biclosed ✔

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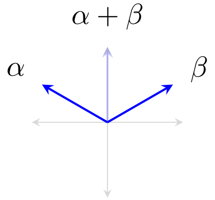

$\{\alpha, \beta\}$ isn't closed ✘

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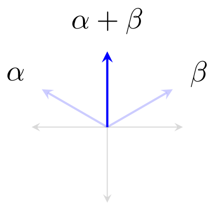

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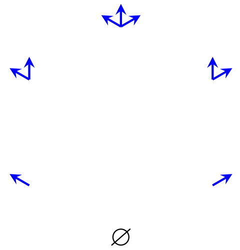

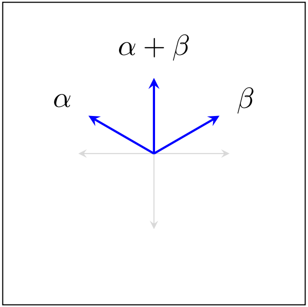

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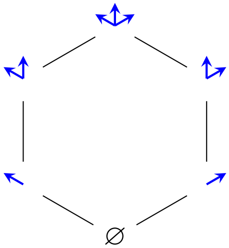
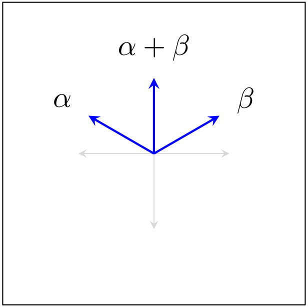
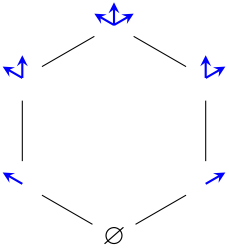


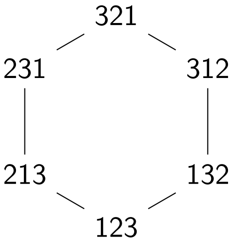


A familiar poset?

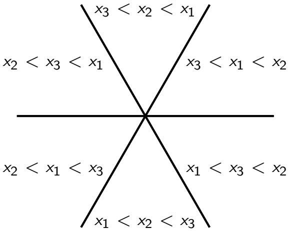
Biclosed sets in Φ^+



The **weak order** on S_3



The **braid arrangement**



Root systems

Fix a symmetric bilinear form $(-,-)$ on V. If $\alpha \in V$ is such that $(\alpha,\alpha) > 0$, then the **reflection** over α is the linear map

$$t_\alpha : V \to V$$
$$\beta \mapsto \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha.$$

Let $\alpha_1, \ldots, \alpha_n$ be a basis for V (call these **simple roots**). We assume $(\alpha_i, \alpha_i) > 0$.

Definition

A subset Φ of V is called a (real) **root system** if the following hold:

1. $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$
2. If $\alpha, \beta \in \Phi$, then $(\alpha,\alpha) > 0$ and $t_\alpha\beta \in \Phi$.
3. If $\alpha = \sum_{i=1}^n a_i\alpha_i \in \Phi$, then either $a_i \geq 0$ for all i, or $a_i \leq 0$ for all i.

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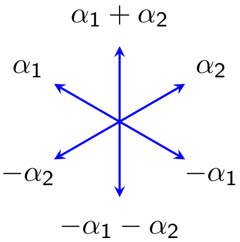

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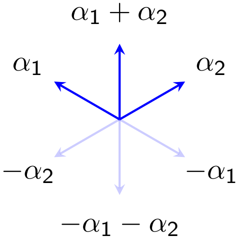

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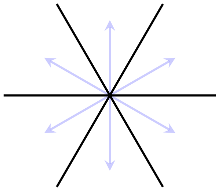

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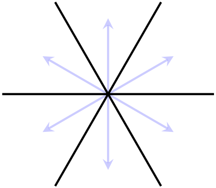

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The **Weyl group** W is the subgroup of $GL(V)$ generated by $\{t_\alpha : \alpha \in \Phi\}$.

Definition

The **Weyl group** W of Φ is the subgroup of $GL(V)$ generated by $\{t_\alpha : \alpha \in \Phi\}$.

The reflections $S = \{t_{\alpha_1}, \ldots, t_{\alpha_n}\}$ already generate W; they are called **simple generators**. The pair (W, S) (turns out to be) a **Coxeter system**.

Definition

Given $w \in W$, the **inversion set** of w is

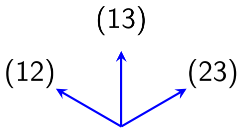
$$N(w) := \{\alpha \in \Phi^+ \mid w^{-1}\alpha \notin \Phi^+\}.$$

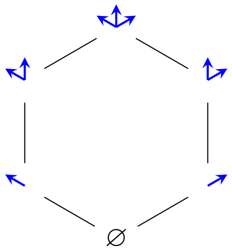
The **weak order** on W is the partial order such that $v \leq w$ iff $N(v) \subseteq N(w)$.

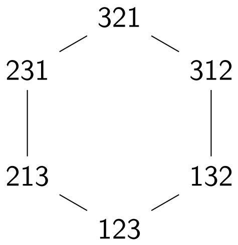
The weak order on S_3

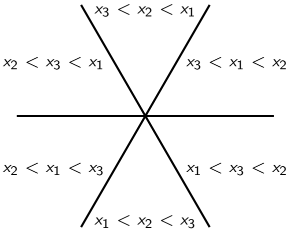
The Weyl group of the A_2 root system is S_3.

Positive roots $\Leftrightarrow$ Reflections $\Leftrightarrow$ Transpositions in S_3



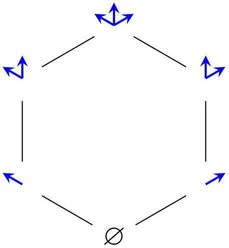


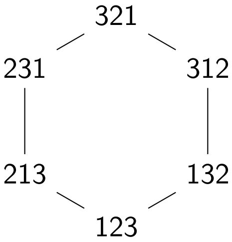


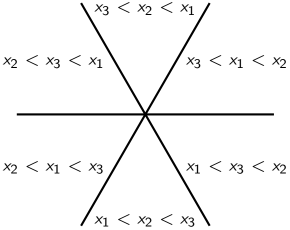


Theorem (Dyer [2])

If Φ is a root system, then the inversion sets $N(w)$ for $w \in W$ are exactly the *finite* biclosed sets in Φ^+. If Φ is finite, then these are in bijection with the regions of the Coxeter arrangement.



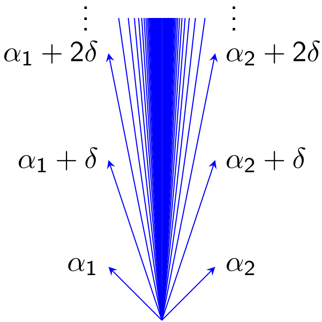




Example: The type $\widetilde{A}_1$ root system

Set $\delta = \alpha_1 + \alpha_2$. The following is a root system[1], the "affine A_1 root system".

$$\Phi = \{\alpha_1 + n\delta, \alpha_2 + n\delta \mid n \in \mathbb{Z}\}.$$



The Weyl group of Φ is isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2$, generated by the two elements $a := t_{\alpha_1}$ and $b := t_{\alpha_2}$.

[1] using a bilinear form such that $(\alpha_1, \alpha_2) = -\sqrt{(\alpha_1, \alpha_1)(\alpha_2, \alpha_2)}$

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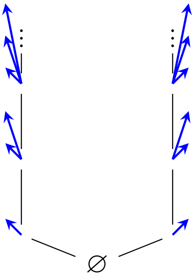
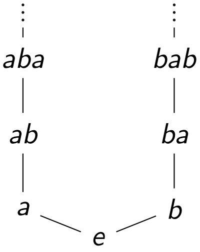
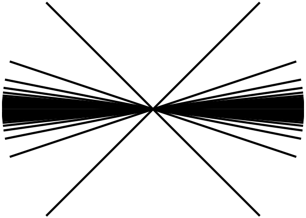
Finite biclosed sets for Φ



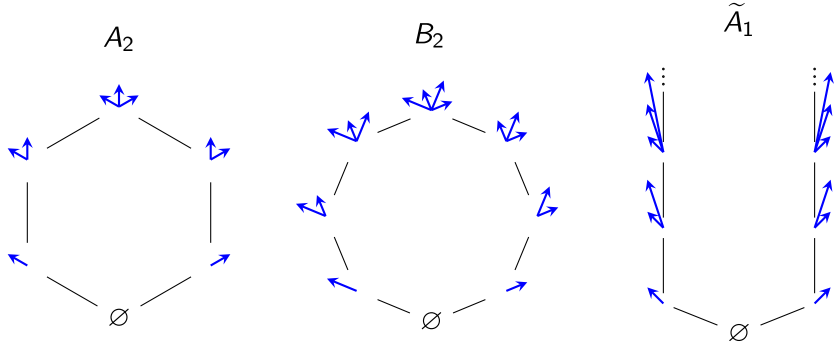
Weak order on $\mathbb{Z}/2 * \mathbb{Z}/2$



Coxeter arrangement for Φ



Comparison of weak orders



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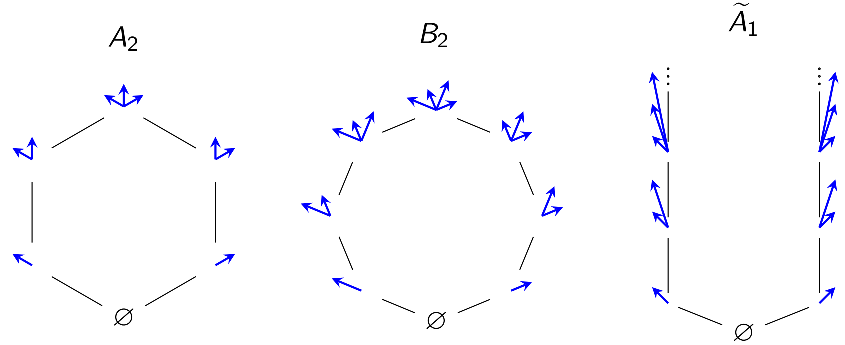

Theorem (Björner–Edelman–Ziegler, Björner)

Weak order on W is a lattice if and only if W (or Φ) is finite. In general, weak order is a **meet-semilattice**.

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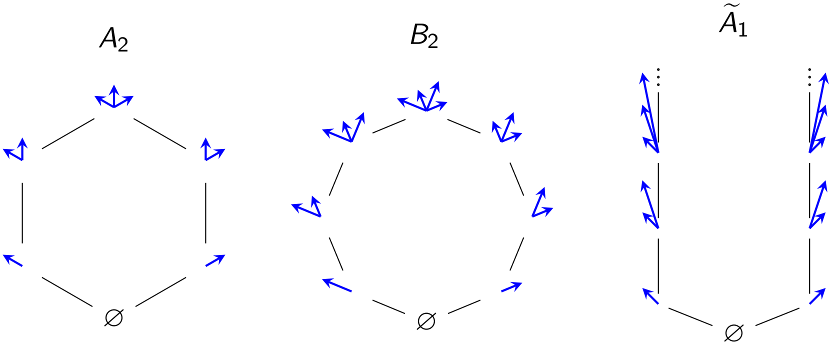


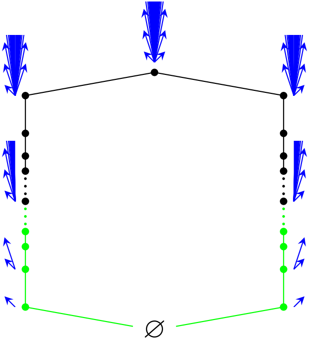
Theorem (Björner–Edelman–Ziegler, Björner)

Weak order on W is a lattice if and only if W (or Φ) is finite. In general, weak order is a **meet-semilattice**.

What happened to the top half of the Hasse diagram for $\widetilde{A}_1$?

Definition

The **extended weak order** on W is the poset of *all* biclosed sets under inclusion.



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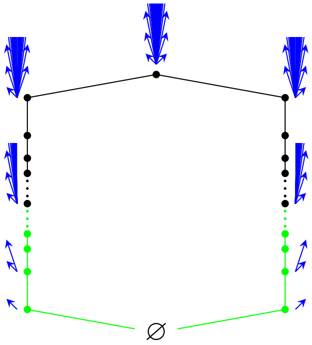

Now we have a lattice! Even a complete lattice!

Is extended weak order always a lattice?

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Conjecture (Dyer [2])

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One of our goals is to solve this conjecture.

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For any root system Φ, extended weak order is a complete lattice.

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Theorem (Wang [3])

The rank 3 affine root systems ($\widetilde{A}_2, \widetilde{C}_2, \widetilde{G}_2$) satisfy Dyer's conjecture.

Results

An **affine root system** Φ is a root system where there is a unique nonzero vector $\delta \in V$ (up to rescaling) such that $(\delta, \delta) = 0$. This is the "least infinite" type of infinite root system. If Φ is affine, then there is a finite root system Φ_0 such that

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Theorem (B.–Speyer)

When Φ is any affine root system, Dyer's conjecture is true: the extended weak order of W is a complete lattice.

A sample

Let S_∞ denote the group of permutations of $\mathbb{Z}$ which fix all but finitely many points. This is the Weyl group of a root system Φ in a countably-infinite dimensional space.

The weak order on S_∞ is not a complete lattice.

Example issue: Joining

$$\pi_0 = \ldots, -3, -2, -1, \mathbf{0}, 1, 2, 3\ldots$$
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$$\pi_2 = \ldots, -3, \mathbf{0}, -2, -1, 1, 2, 3\ldots$$
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$$\vdots$$

gives us an object in which 0 is inverted with all negative integers. We might expect this to look like

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This isn't a permutation! But it is an ordering of the integers.

The theorems for S_∞

Proposition

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Try this one at home!

Thank you!

Question

In finite Coxeter groups, the weak order is a **semidistributive** lattice (elements have canonical join and meet factorizations into irreducible elements).
Does the analogous statement hold for extended weak order?

Problem

Use the extended weak order to construct extended **Cambrian lattices** and **frameworks** for affine type cluster algebras.

Conjecture (Dyer [2])

If B is a biclosed set in Φ_+, then there is a **reflection order** $\prec$ on Φ_+ such that B is an order ideal for $\prec$.

And of course, it remains to answer Dyer's lattice conjecture for any Coxeter group!

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