The hypersimplex and the $m = 2$ amplituhedron

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joint work with Matteo Parisi and Lauren Williams
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Totally nonnegative Grassmannian

$0 < k < n$, $[n] := \{1, \ldots, n\}$, $\binom{[n]}{k} = \{I \subset [n] : |I| = k\}$.

- $Gr_{k,n} := \{V \subset \mathbb{R}^n : \dim V = k\}$
- $V \in Gr_{k,n} \rightsquigarrow$ full rank $k \times n$ matrix A whose rows span V
- $I \in \binom{[n]}{k}$. *Plücker coordinate* $P_I(V) =$ max'l minor of A located in column set I.
- Lusztig, Postnikov: *Totally nonnegative (TNN) Grassmannian*

$$Gr_{k,n}^{\geq 0} = \{V \in Gr_{k,n} : P_I(V) \geq 0 \text{ for all } I\}.$$

- Choose $\mathcal{M} \subset \binom{[n]}{k}$.

$$S_{\mathcal{M}} = \{V \in Gr_{k,n}^{\geq 0} : P_I(V) > 0 \text{ if and only if } I \in \mathcal{M}\}.$$

If $S_{\mathcal{M}} \neq \emptyset$, $\mathcal{M}$ is a *positroid* and $S_{\mathcal{M}}$ is a *positroid cell.*

$$Gr_{k,n}^{\geq 0} = \bigsqcup_{\text{positroids}} S_{\mathcal{M}}$$

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Overview

Moment map:

$$Gr_{k+1,n}^{\geq 0} \xrightarrow{\mu} \mathbb{R}^n$$
$$V \longmapsto \frac{1}{a}\sum |P_I(V)|^2 e_I$$

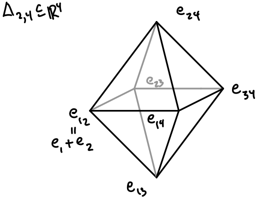
$m = 2$ amplituhedron map:

$$Gr_{k,n}^{\geq 0} \xrightarrow{\tilde{Z}} Gr_{k,k+2}$$
$$[A] \longmapsto [AZ]$$

- Hypersimplex

$$\Delta_{k+1,n} := \mu(Gr_{k+1,n}^{\geq 0})$$

e.g.



- Amplituhedron $\binom{\text{Arkani-Hamed}}{\text{Trnka}}$

$$\mathcal{A}_{n,k,2}^{Z} := \tilde{Z}(Gr_{k,n}^{\geq 0})$$

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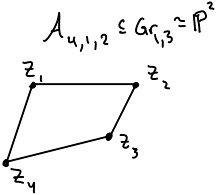

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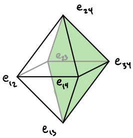
- Positroid polytope

$$\Gamma_{\mathcal{M}} := \mu(\overline{S_{\mathcal{M}}})$$



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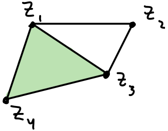

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$$\begin{array}{ccc} Gr^{\geq 0}_{k+1,n} & \xrightarrow{\mu} & \mathbb{R}^n \\ V & \longmapsto & \frac{1}{a}\sum |P_I(V)|^2 e_I \end{array}$$

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- Hypersimplex tile: full-dim'l $\Gamma_{\mathcal{M}}$ s.t. μ injective on $S_{\mathcal{M}}$

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- Positroid "tilings"

$$\Delta_{k+1,n} = \bigcup \Gamma_{\mathcal{M}}$$

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Motivation

Conjecture (LPW '20)

1. *There is a bijection ("T-duality")*

$$\Gamma_{\mathcal{M}} \mapsto Z_{\widehat{\mathcal{M}}}$$

   between positroid tiles for $\Delta_{k+1,n}$ *and positroid tiles for* $\mathcal{A}^Z_{n,k,2}$.
2. $\{\Gamma_{\mathcal{M}}\}$ *a positroid tiling of* $\Delta_{k+1,n} \iff \{Z_{\widehat{\mathcal{M}}}\}$ *a positroid tiling of* $\mathcal{A}^Z_{n,k,2}$ *for all* Z.

Results

Theorem (Parisi–SB–Williams '21)

1. *"T-duality" is a bijection*

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We also give

- Inequality description of tile $Z_{\widehat{\mathcal{M}}}$ which parallels inequality description of T-dual tile $\Gamma_{\mathcal{M}}$.
- Decomposition of $\mathcal{A}^{Z}_{n,k,2}$ into chambers enumerated by Eulerian numbers, "T-dual" to classical triangulation of $\Delta_{k+1,n}$.

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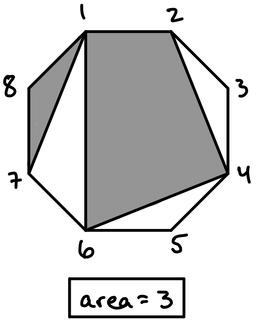
Positroid tiles for $\mathcal{A}^Z_{n,k,2}$

Theorem (PSBW)

1. *Positroid tiles are in bijection with properly bicolored subdivisions of an n-gon* P_n *with area* k*.*
2. *Each non-crossing diagonal gives a linear inequality satisfied by* $\Gamma_{\mathcal{M}}$ *and* $Z_{\widehat{\mathcal{M}}}$*.*



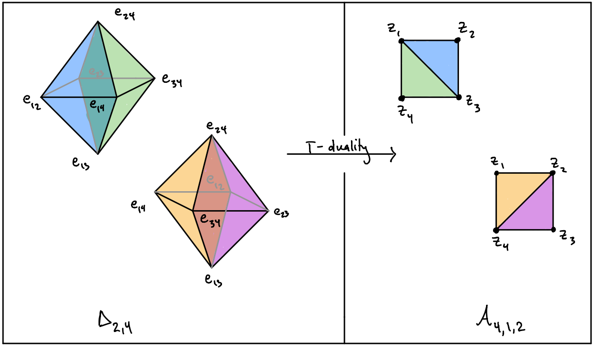
$n \backslash k$	0	1	2	3	4	5
2	1					
3	1	1				
4	1	4	1			
5	1	10	10	1		
6	1	20	48	20	1	
7	1	35	161	161	35	1

# of positroid tiles for $\mathcal{A}^Z_{n,k,2}$ and $\Delta_{k+1,n}$

Positroid tilings of $\mathcal{A}_{n,k,2}^{Z}$ from $\Delta_{k+1,n}$

Theorem (PSBW '20)

$\{\Gamma_{\mathcal{M}}\}$ *a positroid tiling of* $\Delta_{k+1,n} \iff \{Z_{\widehat{\mathcal{M}}}\}$ *a positroid tiling of* $\mathcal{A}_{n,k,2}^{Z}$ *for all* Z.



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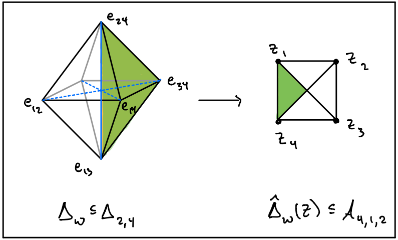
Proof technique: take simultaneous refinement of all positroid tilings on both sides and match up the pieces.



Thanks for listening!

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T-duality

Positroids of type (k, n) are in bijection with *(decorated) permutations*:

$$C = \begin{bmatrix} | & | & & | \\ C_1 & C_2 & \dots & C_n \\ | & | & & | \end{bmatrix} \in S_{\mathcal{M}}.$$

$$\pi_{\mathcal{M}} : i \mapsto j$$

where C_j is the first column such that $C_i \in span\{C_{i+1}, \dots, C_j\}$.

T-duality:

$$\pi = \begin{matrix} 1 & 2 & & n \\ \downarrow & \downarrow & \dots & \downarrow \\ a_1 & a_2 & & a_n \end{matrix} \quad \mapsto \quad \hat{\pi} = \begin{matrix} 1 & 2 & & n \\ \downarrow & \downarrow & \dots & \downarrow \\ a_n & a_1 & & a_{n-1} \end{matrix}$$

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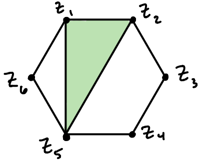
Examples of $m = 2$ amplituhedra

Z a $n \times (k+2)$ matrix with positive max'l minors.

$$\tilde{Z}: \; Gr_{k,n}^{\geq 0} \longrightarrow Gr_{k,k+2} \qquad \mathcal{A}_{n,k,2}(Z) := \tilde{Z}(Gr_{k,n}^{\geq 0})$$
$$[A] \longmapsto [AZ]$$

- $k + 2 = n$: $\mathcal{A}^Z_{n,k,2} \cong Gr_{k,n}^{\geq 0}$. Only one positroid tile.
- $k = 1$: $\mathcal{A}^Z_{n,k,2}$ = cyclic polygon with n vertices in $\mathbb{P}^2$. Positroid tiles are triangles.



$\mathcal{A}_{6,1,2} \subseteq \mathbb{P}^2$

- (Bao–He): Give tilings of $\mathcal{A}^Z_{n,k,2}$ via recursion.

Finer decomposition

- (Stanley, Sturmfels, Lam–Postnikov): Triangulation of $\Delta_{k+1,n}$ with max'l simplices indexed by permutations of $n-1$ with k descents.

$$\Delta_{k+1,n} = \bigcup \Delta_w$$

- Simultaneous refinement of all positroid tilings gives

$$\mathcal{A}^Z_{n,k,2} = \bigcup \hat{\Delta}^Z_w$$

and as long as $\hat{\Delta}^Z_w \neq \emptyset$,

$$\Delta_w \subset \Gamma_{\mathcal{M}} \iff \hat{\Delta}^Z_w \subset Z_{\hat{\mathcal{M}}}.$$

- Problem: $\hat{\Delta}^Z_w$ may be empty for some Z.
- However, for any w, can find Z so that $\hat{\Delta}^Z_w \neq \emptyset$.