A Pattern Avoidance Characterization for Smoothness of Positroid Varieties

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Based on joint work with:
Jordan Weaver
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Outline

Motivation: Pattern avoidance on decorated permutations

Positroid varieties

Characterizing Smooth Positroid Varieties

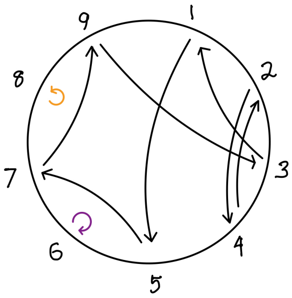
Enumeration

Future Work

Decorated Permutations

Defn. A *decorated permutation* $w^{\circlearrowright}$ on n elements is a permutation $w \in S_n$ together with an orientation clockwise or counterclockwise, denoted $\overrightarrow{i}$ or $\overleftarrow{i}$ respectively, assigned to each fixed point of w.

$$w^{\circlearrowright} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 1 & 2 & 7 & \overrightarrow{6} & 9 & \overleftarrow{8} & 3 \end{bmatrix} = 54127\,\overrightarrow{6}\,9\,\overleftarrow{8}\,3$$



Decorated Permutations

Sylvie Corteel studied the q-analogs of Eulerian numbers related to the number of alignments and weak exceedances of permutations, which come from the enumeration of totally positive Grassmann cells following work of Williams and Postnikov.

Question. (Corteel, 2006) Can we define generalized patterns for decorated permutations?

Question.(Weaver, 2018) Are there interesting families of decorated permutations avoiding certain generalized patterns?

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Question.(Weaver, 2018) Are there interesting families of decorated permutations avoiding certain generalized patterns?

Answer. Yes! The ones indexing smooth positroid varieties.

Grassmannian Varieties

Notation.

- Fix $0 \leq k \leq n$.
- Mat_{kn} = the set of full rank $k \times n$ matrices over $\mathbb{C}$.
- $Gr(k,n)$ = *Grassmannian variety* of k-dim subspaces in $\mathbb{C}^n$
- $Gr(k,n) \approx \mathrm{GL}_k \backslash \mathrm{Mat}_{kn}$.
- $\Delta_J(M)$ = determinant of the $[k] \times J$ minor of M for $J \in \binom{[n]}{k}$.

The Grassmannian varieties are smooth manifolds via the *Plücker coordinate* embedding $M \mapsto (\Delta_J(M) : J \in \binom{[n]}{k})$ of $Gr(k,n)$ into projective space, including the case $k = n = 0$.

Plücker coordinates

$$\begin{bmatrix} 0 & 3 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \mapsto [0:0:0:0:0:0:3:6:3:1:2:1:0:2:4] \in \mathbb{P}^{15}$$

Nonvanishing coordinates at exactly the coordinates

$$\{\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,6\},\{5,6\}\} \subseteq \binom{[6]}{2}.$$

Def. The *matroid* of $A \in \mathrm{Mat}_{kn}$ is determined by the set of *bases*

$$\mathcal{M}_A = \{J \in \binom{[n]}{k} \mid \Delta_J(A) \neq 0\}.$$

or by the *non-bases*

$$\mathcal{Q}_A = \{J \in \binom{[n]}{k} \mid \Delta_J(A) = 0\}.$$

Positroids

Defn. (Postnikov) A *positroid* of rank k on ground set $[n]$ is a matroid of the form M_A for a matrix $A \in \mathrm{Mat}_{kn}$ such that every nonzero Plücker coordinate $\Delta_J(A)$ is positive.

Thm. (Postnikov 2006 + Oh 2011) There are bijections

1. positroids $\mathcal{M}$ of rank k on a ground set of size n,
2. decorated permutations w° on n elements with k anti-exceedances,
3. Grassmann necklaces $\mathcal{N} = (I_1, \dots, I_n) \in \binom{[n]}{k}^n$, and
4. Grassmann intervals $[u, v]$ in $Gi(k, n)$.

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Important in the theory of the totally positive part of the Grassmannian variety, cluster algebras, and soliton solutions to the KP equations and have connections to statistical physics, integrable systems, and scattering amplitudes.

See Williams.ICM.2022, along with Lusztig.1994, Rietsch.1998, Fomin-Zelevinsky.2002, Kodama-Williams.2015, AHBCGPT.2016
Positroids are closed under restriction, contraction, duality, and

Positroid Varieties

Defn.(Rietsch, Postnikov, Knutson-Lam-Speyer) Given a decorated permutation $w^{\circ} \in S_{n,k}^{\circ}$ along with its associated Grassmann interval $[u, v]$ and positroid $\mathcal{M} = \mathcal{M}(w^{\circ}) \subseteq \binom{[n]}{k}$, the *positroid variety* $\Pi_{w^{\circ}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$ is the subvariety of $Gr(k, n)$ with vanishing ideal generated by the Plücker coordinates indexed by the nonbases of $\mathcal{M}$, $\{\Delta_J : J \notin \mathcal{M}\}$.

Thm.(Knutson-Lam-Speyer) The positroid variety $\Pi_{[u,v]}$ is the projection of the Richardson variety $X_u^v \subseteq \mathcal{F}\ell(n)$ to $Gr(k, n)$, so $\Pi_{[u,v]} = \pi_k(X_u^v)$.

Positroid Varieties

The vanishing coordinates of

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \mapsto [0:0:0:0:0:0:3:6:3:1:2:1:0:2:4]$$

are exactly the coordinates indexed by the nonbases of $\mathcal{M}_A$

$$\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{4,5\}\} \subseteq \binom{[6]}{2}.$$

Therefore, the points in the positroid variety $\Pi_{\mathcal{M}_A}$ are represented by the full rank complex matrices of the form

$$\begin{bmatrix} 0 & a_{12} & ca_{12} & a_{14} & da_{14} & a_{16} \\ 0 & a_{22} & ca_{22} & a_{24} & da_{24} & a_{26} \end{bmatrix}.$$

Smooth versus singular varieties

Def. A variety X is *singular* at a point $x \in X$, if the dimension of the tangent space to X at x is strictly larger than the dimension of X. If not such point exists, X is *smooth*.

Def. If $X = Var(f_1, \dots, f_s)$, the *Jacobian matrix*, *Jac*, is the matrix of partial derivatives of the f_i at each of the possible variables.

Def. The $\operatorname{rank}(Jac|_x)$ is the codimension of the tangent space to X at the point x.

Characterizing Smooth Positroid Varieties

Thm. (Billey-Weaver 2022) Let $\mathcal{M}$ be a rank k positroid on $[n]$ with associated decorated permutation $w^{\circlearrowleft}$, and Grassmann interval $[u, v]$. Then, the following are equivalent.

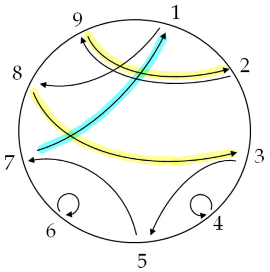
1. The positroid variety $\Pi_{w^{\circlearrowleft}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$ is smooth.
2. The decorated permutation $w^{\circlearrowleft}$ has no crossed alignments.
3. The chord diagram $D(w^{\circlearrowleft})$ is a disjoint union of spirographs.
4. The positroid $\mathcal{M}$ is a direct sum of uniform matroids.

Some Patterns in Decorated Permutations

Patterns.

- *alignments*: two directed edges in same direction
- *misalignments*: two directed edges in opposite direction
- *crossings*: two edges that must have a common point
- *crossed alignment*: an alignment plus a third edge crossing both sides

Consider the decorated permutation $w^{\circ} = 895\,\overleftarrow{4}\,7\,\overrightarrow{6}\,132$

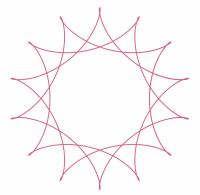


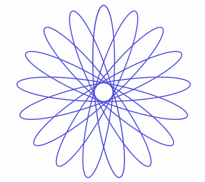
$(9 \mapsto 2, 8 \mapsto 3)$ highlighted in yellow is an *alignment*,

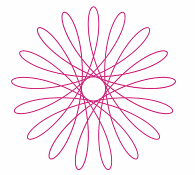
Patterns in Decorated Permutations

Patterns.

- *spirograph*: chord diagram of a connected decorated perm with $w(i) \mapsto i + m$ for all $i \in [n]$.







Thx to "Spirographs" made by the Spirograph Maker app for the iphone.

Decorated Permutations to Grassmann intervals

Defn. The *anti-exceedance set* of w° is

$$I_1(w^{\circ}) = \{i \in [n] \mid i < w^{-1}(i) \text{ or if } w^{\circ}(i) = \overrightarrow{i}\}$$

If $i \in [n]$ is not an anti-exceedance, it is an *exceedance*. Let $S^{\circ}_{n,k}$ be the set of decorated permutations on n elements with k anti-exceedances.

$$w^{\circ} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 1 & 2 & 7 & \overrightarrow{6} & 9 & \overleftarrow{8} & 3 \end{bmatrix} \Longrightarrow I_1(w^{\circ}) = \{1, 2, 6, 3\}.$$

Shuffling the anti-exceedances to the front bijectively determines the *Grassmann interval* $[u, v]$ associated to $w^{\circ} \in S^{\circ}_{n,k}$, where $u \le v$ in Bruhat order on S_n and v is k-Grassmannian, so $v_1 < v_2 < \cdots < v_k$ and $v_{k+1} < \cdots < v_n$.

$$\begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 3 & 4 & 6 & 9 & 1 & 2 & 5 & 7 & 8 \\ 1 & 2 & 6 & 3 & 5 & 4 & 7 & 9 & 8 \end{bmatrix}.$$

Decorated Permutations to Positroids

The *Grassmann necklace* for $w^{\circlearrowleft} = 54127\,\overrightarrow{6}\,9\,\overleftarrow{8}\,3$ is

$$(I_1, \ldots, I_9) = (\{1236\}, \{2356\}, \{3456\}, \{1456\}, \{1256\}, \{1267\}, \{1267\}, \{1269\}, \{1269\}).$$

The *positroid* for $w^{\circlearrowleft}$ is

1236	1246	1256	1267	1269	1346	1356
1367	1369	1456	1467	1469	2356	2367
2369	2456	2467	2469	3456	3467	3469

Here, for each $r \in [n]$, let $r <_r <_r r+1 < \cdots n <_r 1 <_r < \cdots <_r r-1$,

$$I_r(w^{\circlearrowleft}) = \{i \in [n] \mid i <_r w^{-1}(i) \text{ or if } w^{\circlearrowleft}(i) = \overrightarrow{i}\},$$

$$\mathcal{M}(w^{\circlearrowleft}) := \left\{ I \in \binom{[n]}{k} : I_r(w^{\circlearrowleft}) \leq_r I \text{ for all } r \in [n] \right\}$$

Commutative Diagram of Bijections

$$\begin{tikzcd}
S^{\circ}_{n,k} \arrow[rr, "shuffle"] \arrow[d, "necklace"] & & Gi(k,n) \arrow[d, "initial"] \\
\text{Grassmann necklaces in } \binom{[n]}{k}^n \arrow[rr, "shiftedGale"] & & \text{Positroids in } \binom{[n]}{k}
\end{tikzcd}$$

Grassmann intervals to Positroids via *initial sets*.

$$\mathcal{M} = \{y[k] \ : \ y \in [u, v]\}.$$

Outline of Proof

Thm. [Postnikov, KLS]

$$\mathrm{codim}(\Pi_{w^{\circlearrowleft}}) = \#\mathit{Alignments}(w^{\circlearrowleft}) = k(n-k) - [\ell(v) - \ell(u)].$$

Corollary. A positroid variety $\Pi_{w^{\circlearrowleft}}$ is a singular at x if and only if

$$\mathrm{rank}(Jac|_x) \; < \; \mathrm{codim}\,\Pi_{w^{\circlearrowleft}} \; = \; \#\mathit{Alignments}(w^{\circlearrowleft}).$$

Outline of Proof

By Construction.: For every $A \in \Pi_{\mathcal{M}}$, the first nonzero Plücker coordinate in lex order of A is in $\mathcal{M}$. For every $J \in \mathcal{M}$, the matrix $A_J \in \Pi_{\mathcal{M}}$ which is the identity in columns J.

Thm. Assume $A \in \Pi_{\mathcal{M}}$ and $J \in \mathcal{M}$ indexes its first nonzero Plücker coordinate in lex order. Then the codimension of the tangent space to $\Pi_{\mathcal{M}}$ at A is bounded below by

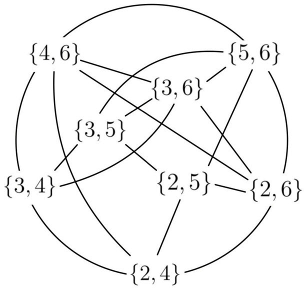
$$\mathrm{rank}(Jac|_{A_J}) = = \#\left\{ I \in \binom{[n]}{k} \setminus \mathcal{M} \;:\; |I \cap J| = k-1 \right\}.$$

Hard Step.: Given a starboard tacking crossed alignment based at 1, the set on RHS maps injectively into the set of alignments of $w^{\circlearrowleft}$ and for $J = I_1(w^{\circlearrowleft}) \in \mathcal{M}$ there is at least one alignment that is not in the image.

Johnson Graphs

The *Johnson graph* $J(k,n)$ has vertices labeled by k-subsets of $[n]$ and two k-subsets I, J are connected by an edge precisely if $|I \cap J| = k - 1$.

For a positroid $\mathcal{M} \subseteq \binom{[n]}{k}$, let $J(\mathcal{M})$ denote the induced subgraph of the Johnson graph on the vertices in $\mathcal{M}$. We call $J(\mathcal{M})$ the *Johnson graph of* $\mathcal{M}$.



Regular Johnson Graphs and Smooth Positroid Varieties

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1. The positroid variety $\Pi_{w^{\circ}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$ is smooth.
2. The decorated permutation w° has no crossed alignments.
3. The chord diagram $D(w^{\circ})$ is a disjoint union of spirographs.
4. The positroid $\mathcal{M}$ is a direct sum of uniform matroids.
5. The graph $J(\mathcal{M})$ is regular, and each vertex has degree $\ell(v) - \ell(u)$.

Enumerative Results

Similar to Ardila-Rincón-Williams enumeration of positroids via connected positroids, we use a theorem of Speicher to enumerate all smooth positroids via connected smooth positroids (aka spirograph permutations).

Thm. The number of smooth positroids on ground set $[n]$ is the coefficient

$$s(n) = [x^n]\frac{1}{n+1}\left(1 + 2x + \sum_{i=2}^{n}(i-1)x^i\right)^{n+1}. \tag{1}$$

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$$\begin{aligned} s(51)/s(50) &\approx 5.4489775, \\ s(101)/s(100) &\approx 5.528236, \\ s(151)/s(150) &\approx 5.555362, \\ s(201)/s(200) &\approx 5.569062, \\ s(251)/s(250) &\approx 5.5773263. \end{aligned}$$

Based on this data, we conjecture $s(n) \approx (c^n)$ for $c \leq 6$.

Enumerative Results

Notation. For $1 \leq k \leq n$, let $b_{0,0} = 1$, and $b_{0,k} = b_{n,0} = 0$ if $n > 0$ or $k > 0$,

- $B_{n,k}(x_1, \ldots, x_{n-k})$ Bell polynomial
- $b_{n,k} = B_{n,k}(2 \cdot 1!, 1 \cdot 2!, 2 \cdot 3!, \ldots, (n-k) \cdot (n-k+1)!)$

Cor. The number of smooth positroids on ground set $[n]$ is

$$s(n) = \sum_{k=1}^{n} \frac{b_{n,k}}{(n-k+1)!}. \tag{2}$$

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Cor. The number of smooth positroids on ground set $[n]$ with k connected components is $\frac{b_{n,k}}{(n-k+1)!}$.

Future Work

Open problems.

1. Study the geometry and equivariant cohomology of a positroid variety via the induced directed Johnson graph on $\mathcal{M}$.
2. Identify the singular locus of a positroid variety.
3. Study the enumeration and coset structure of the group operations of flip, inverse, rotation for perms, or for derangements, or SIF perms.
4. Study the combinatorics of the directed Johnson graphs of positroids.