Pingala and the Beginnings of Combinatorics in India

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Background - the very beginning

Recorded maths in India begins with evidence from the archaeological remains of the Indus Valley (Harappan) civilisation (IVC) ($\sim 2600 - 1800$ BCE). Among the highlights:
- Knowledge of counting up to fairly large numbers.
- Binary ratios dominant in artefacts.
- Indirect evidence for counting in **base 8**.
- Aproximate cube root of 2 (Remember the oracle of Delphi!)

Compare: Earliest written place-value numbers (base 60, no zero) from Mesopotamia, ~ 1800 BCE. There were commercial contacts between IV and Mesopotamia but not after 1900 BCE.

(Interesting geometry: Common chord of intersecting circles is perpendicular to line of centres $\rightarrow$ planar decorative patterns, still popular. But no Pythagoras)

Background - decimal numbers, true Vedic Mathematics

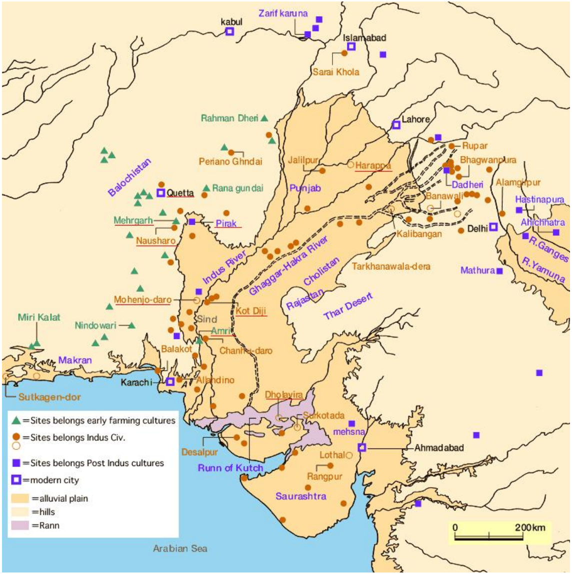
The very first *documented* mathematical knowledge comes from a non-math (religious) text, *Ṛgveda* ($\sim$ 1200 BCE): $>$ 1000 names of numbers showing (through **grammatical rules**) a total mastery of the principles of number formation in a base (e.g. 10), as in "two thousand and twenty two". Standard math. histories didn't know this. (For them decimal numbers are necessarily a string of **written** symbols, e.g "2022"). The reason for the Vedic popularity of such a nominal or, in modern terminology, polynomial representation of numbers rather than a symbolic or coefficient representation:

Background - the oral tradition

The Vedic people did not have writing. Everything had to have an uttered name. Texts had to be memorised and recited. Hence the popularity of versification – a regular, repetetive sonic structure – as a mnemonic device,

No need for a place-value zero (just drop it as in "two thousand and twenty two") in the nominal representation. Contrast with 2022 ($\neq$ 222). This was a problem in Old Babylonian. But 0 as a number on its own (the cardinality of the empty set) is a different story.

Background - the geography



Background - Pāṇini ($\sim$ 5th C. BCE)

Author of the definitive grammar of the Sanskrit of his time (and of all time to come). Turned grammar into set theory: categorised linguistic units – syllables, words, . . . — into sets having a common function, or having a phonetic similarity (even gave the sets names, metalinguistic markers), and reduced grammar ("the first science of India") to the study of maps between sets and their composition. The final product is intricate but unambiguous.

– Enumerative Vs *categorical*: Patañjali's fable.
– Pāṇini's techniques are best exemplified in the formation of number names, there are strict rules (famous scholars have got them wrong).

Our hero Pingala (3rd - 2nd C. BCE) brought Paninian categorical methods to the mathematical classification of metres in his work *Chandaḥsūtra*, "Aphorisms on Prosody". Much simpler (finite) problem, complete solution.

Pingala in his times

Like Pāṇini, Pingala very likely belonged to NW India (map), at that time one of the civilisational crossroads of the antique world. In fact, beginning around the 5th C. BCE, all of N India experienced a period of intellectual ferment. Some high points of interest to us (apart from Pāṇini):

- Writing (in a syllabic script, Brahmi) and inscribed symbolic numbers make their appearance. May have influenced Pingala.
- Negative numbers.
- Exploration of larger and larger numbers ($\rightarrow \infty$) (Patanjali's fable), including in literarary and religious works.

What is a metre?

- In Sanskrit every syllable belongs to one of two subsets, depending on their duration (*mātra**), labelled l and g (by Pingala himself). Enables a categorical statement of his main propositions. $\mathbf{g} \in g$ has twice the duration of $\mathbf{l} \in l$.
- A stanza has normally 2, 3 or 4 lines or feet (*pāda**).
- A line is a sequence of a fixed number of syllables, its length; i.e., a line of length n is a sequence $m_i = x_1 x_2 \cdots x_n$ of syllables in which each x_j assigned one of two values $\mathbf{l}$ or $\mathbf{g}$.

For Pingala, a metre is such a sequence of $\mathbf{l}$ and $\mathbf{g}$. The poets of course had no use for the vast majority of such 'mathematical' metres. One suspects poetry was just an excuse for the maths. (The metre as applied to a stanza is a set of sequences, one for each line; may or may not be the same).

The 5 propositions

of Pingala are answers to the following questions:

1. Make a canonically ordered list $\{m_i\}$ of all metres of length n.

2 and 3. Given the rule of prop. 1, what is m_i for a given index i and conversely (without writing out the whole list)?

4. Find an efficient algorithm for the number $N(n)$ of sequences of length n.

5. What is the number $N(n, k)$ of sequences (metres $m(n, k)$) with a given number $k < n$ of **l** (or **g**)?

The answers are in a cryptic (even more than in Pāṇini) *sūtra* form. Need later commentaries to interpret them.
[Jayant Shah's comparative analysis of these commentaries: "A History of Pingala's Combinatorics", Gaṇita Bhāratī **35**, 1 (2013), is an excellent guide].

Proposition 1 – the ordering

Several different (obvious) rules possible. The one chosen: **g** first, adjoin on left (the number in brackets is the index i).

- Length = 1: (1) **g**, (2) **l**.
- Length = 2: (1) **gg**, (2) **lg**, (3) **gl**, (4) **ll**.
- Length = 3: (1) **ggg**, (2) **lgg**, (3) **glg**, (4) **llg**. (5) **ggl**, (6) **lgl**, (7) **gll**, (8) **lll**.

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To get *all* sequences of length n without duplication, adjoin **g** and **l** in order to the left of the sequences (in order) in the list for length $n - 1$.

Props. 2 and 3 are in relation to this indexing.

Binary numbers? **No**

It has been claimed that Pingala's lists show an awareness of binary numbers. As given above, neither choice **l** $= 0$, $\mathbf{g} = 1$ or $\mathbf{g} = 0$, **l** $=$ 1 turns a list into binary numbers in order. Some variant orderings do, e.g., set (**l,** $\mathbf{g}) = (1, 0)$ and read each sequence backwards. Not surprising in combinatorics involving two-valued variables.

Historically, place-value symbolic numbers and the symbolic 0 were 600 - 700 years in the future (Bakhshali manuscript). The base was always 10 in India, including presumably in Pingala's indexing – otherwise no point in

Propositions 2 and 3

Any binary count variant of the list affords a simple association: index $\rightarrow$ metre. The ith entry in the list for any length n corresponds to the binary representation of $i-1$ (1st entry is 0). So, in the variant above, express $i-1$ in binary form, adjoin zeros on the left to bring the length up to n, read it backward and substitute 0 and 1 by **g** and **l**. Example; the 3rd metre of length 4:

$$2 \rightarrow 10 \rightarrow 0010 \rightarrow 0100 \rightarrow \mathbf{glgg}$$

Pingala's way (no explicit binary numbers): If i is odd, put down **g** and replace i by $i' = (i+1)/2$; if i is even, put down **l** and replace i by $i' = i/2$. Repeat with i', putting down the new *mātra* to the **right** of the first *mātra*. Continue till the string of updated *mātra*s has length n. Example: $n = 4, i = 3$:

$$i = 3\ (\mathbf{g}) \rightarrow i' = 2\ (\mathbf{gl}) \rightarrow i'' = 1\ (\mathbf{glg}) \rightarrow i''' = 1\ (\mathbf{glgg})$$

Props. 2 and 3: **recursion**

[Easy to prove the correctness of Pingala's algorithm. Prop. 3 just reverses the steps of prop. 2], omitted.

More than the result, the interest is in the method:

- Elementary but original and sophisticated, 'modern'.
- Use of recursion as a mathematical tool, a 'first' apart from based (decimal) numbers themselves: the list is constructed recursively; props. 2 and 3 are demonstrated recursively: "a procedure consisting of a chain of steps in which the output of a given step is the input in the next step", widely prevalent in Indian maths (and culture generally). In fact
- All of Pingala's statements are recursively generated (cf. Pāṇini's "generative grammar"). More, some are examples of recursive **descent**: reduce a problem to a simpler one (involving smaller numbers) in recursive steps till it becomes trivial. Formal mathematical induction (**ascent**?) came much later (Mādhava $\sim$ 1400 CE).

Prop. 4: computational economy?

We know $N(n) = 2^n$. Pingala knew too, from the construction of the list: $n \to n+1$ results in the **recursion relation** $N(n+1) = 2N(n)$. Once again, the interest is in the algorithm.

Write $N(n)$ as $(N(n/2))^2$ if n is even, $2(N(n-1)/2)^2$ if n is odd, and iterate. Two extreme examples:

$$N(8) = N(4)^2 = N(2)^4 = N(1)^8 = 2^8$$

$$N(7) = 2N(3)^2 = 2(2N(1)^2)^2 = 2^{1+2+4}$$

Why this fancy factorisation?

The (numerical) **zero** makes its debut

So, once again, recursive descent to the fore. Two branchings in the recursion (as also in prop. 2), depending on whether $N(i)$ at an initial or intermediate i is even or odd. The recursive rules are stated categorically by giving names to the subsets of even and odd N. The names are

$\{\text{even}\} = $ *dvi*

$\{\text{odd}\} = $ *śūnya*

The tags have a literal meaning (unlike in Pāṇini): *dvi** is the number 2 and *śūnya* ('void') can only be the number 0 – its first occurrence, through a side door so to say.

Prop. 5: binomial coefficients

Symmetry: $N(n, n-k) = N(n, k)$.
Initial value: $N(n, 0) = N(n, n) = 1$.

Multiple descent: To get all $m(n, k)$ without duplication, adjoin (on the left, say) one **l** to every $m(n-1, k-1)$ and one **g** to every $m(n-1, k)$:

$$N(n, k) = N(n-1, k-1) + N(n-1, k).$$

We know this is the recurrence relation for **binomial coefficients**. Pingala's solution: iterate on the second term and carry on:

$$N(n, k) = N(n-1, k-1) + N(n-2, k-1) + \cdots + N(k-1, k-1).$$

Next, 'descend' each term on the right. Repeat. Pingala probably started with $N(2, 1) = 2$ and 'induced up'. One can plot the numbers in the (n, k) plane if one likes as was done much later by Halāyudha (10th C. CE) (and Pascal of course).

Pingala's legacy

Pingala's work was elaborated and generalised over the next 1500 years or so. Most notable are theorists of the performing arts – music with its 4 time measures and 7 notes offered a rich field of study:
Bharata ($\sim 0 \pm 100$ CE; treatise on music and dance, still in use).
Virahāṅka (6th - 8th C. CE; combinatorics of rhythm (duration), see below).
Halāyudha (10th C. CE; commentary on Pingala, Pascal triangle).
Śārngadeva (13th C. CE; *Saṅgīta-ratnākara*, magnificent treatise on the science of music. For its maths, far beyond Pingala, see Raja Sridharan *et al.* in the CMI Seminar volume).

Among the 'pure' mathematicians was Nārāyaṇa Paṇḍita (mid 14th C. CE), a true polymath, one of the greats.

I skip this long history, except for

Combinatorics of rhythm: Fibonacci series

Simpest generalisation of Pingala. Define duration of a metre with p **l**s and q **g**s as $p + 2q =: n$ (essential parameter in Indian music: a rhythm cycle (*tāla*) covers a fixed duration). Pingala's indexing props. 1, 2, 3 are unaffected. The analogue of prop. 4 now asks:

How many distinct *tāla*s $D(n)$ in a cycle of duration n?

Given Pingala's list, all sequences of duration n are obtained by adjoining (on the left) a **g** to all sequences of duration $n - 2$ and a **l** to all sequences of duration $n - 1$:

$$D(n) = D(n-2) + D(n-1).$$

This is just the recursion relation for Fibonacci numbers.

[For more, see Parmanand Singh, Hist. Math. **12**, 229 (1985). For other, more complicated sequences of numbers arising in musical theory, see Raja Sridharan *et al.*].

Change of scene: Mādhava's calculus and the sine series

Mādhava($\sim$ 1400 CE)'s invention of calculus for trigonometric functions – divide the variable into a large number n of equal parts, linearise the problem (e.g., rectification) locally, add up the bits and take the limit $n \to \infty$ at the end. Original and rigorous, even discusses convergence. Example: **power series** for sine and cosine. Here is his method in the limit:

- Show the function satisfies the differential equation

$$\frac{d^2 \sin x}{dx^2} = -\sin x.$$

- Turn it into the integral equation

$$\sin x = x - \int dy \int \sin z \; dz$$

- Replace $\sin z$ by its integral representation and iterate.

The sine series: binomial coefficients as 'sums of sums'

Mādhava did all this for finite differences after dividing x into $2n$ equal parts – heavy stuff. The end result is

$$\sin x = \sin \frac{x}{n}\{S_0(n) - 4\sin^2\frac{x}{2n}S_2(n-1) + 4^2\sin^4\frac{x}{2n}S_4(n-2) - \cdots\}.$$

The coefficients S are recursively defined: $S_k(n) = \sum_{i=1}^{n} S_{k-1}(i)$ (sums of sums) with $S_k(0) = 0$ and $S_0(n) = n$.

Immediate: separate the last term: $S_k(n) = S_{k-1}(n) + S_k(n-1)$. Surprise! This is Pingala's recurrence relation for binomial coefficients if we redefine $S_k(n) = N(n+k, n-1)$.

Easy to take the limit $n \to \infty$, with $\sin(x/n)$ replaced by x/n, and get the familiar series.

(Proof withheld – *alpadhiyāṃ hita*. Pingala and prosody not mentioned in texts).

Recursion, descent, induction

Recursive reasoning of which we have seen many examples above is a widely used technique in India's mathematics. Descent methods likewise. Combinatorics is tailor-made for their application.

Recursion in general: 'Guess' a reasonable linear approximation to the solution of a nonlinear problem, compute the error, set up the (nonlinear) equation for the error, linearise, repeat (*śuddhīkaraṇa* = purification, *saṃskāra* = refining). Simplest example: the Bakhshali square root – extremely efficient computationally even in school exercises.

The first formal inductive proofs are from Mādhava's calculus. One of his followers set out their logical basis (1520 - 1525 CE) in the form, essentially, of the succession axiom of Peano. Indians took foundational questions seriously but rejected axiomatism.

Conclusion (in more senses than one)

Mādhava has a place in the pantheon of great mathematicians, only recently beginning to be recognised as the Titan he was. He founded a line of disciples of the highest quality, a school that flourished over 2 centuries ($\sim$ 1400 - 1600) in a group of villages on the coast of Kerala. The end (and the end of the long unbroken tradition of mathematics in India as a whole) came with the arrival of the Portuguese on the Kerala coast. The mathematical villages of the Madhava school in particular saw much violence and bloodshed.