Equalities and inequalities involving Schur polynomials

Apoorva Khare
IISc and APRG (Bangalore, India)

Formal Power Series and Algebraic Combinatorics
July 2022

Schur polynomials

Given a decreasing N-tuple of integers $n_1 > \cdots > n_N \geqslant 0$,
the corresponding **Schur polynomial** over a field $\mathbb{F}$ (say char $\mathbb{F} = 0$) is the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{\mathbf{n}}(u_1, \dots, u_N) := \frac{\det(u_i^{n_j})_{i,j=1}^N}{\det(u_i^{N-j})} = \frac{\det(u_i^{n_j})_{i,j=1}^N}{V(\mathbf{u})}$$

for pairwise distinct $u_i \in \mathbb{F}$.

Schur polynomials

Given a decreasing N-tuple of integers $n_1 > \cdots > n_N \geqslant 0$, the corresponding **Schur polynomial** over a field $\mathbb{F}$ (say char $\mathbb{F} = 0$) is the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{\mathbf{n}}(u_1, \dots, u_N) := \frac{\det(u_i^{n_j})_{i,j=1}^N}{\det(u_i^{N-j})} = \frac{\det(u_i^{n_j})_{i,j=1}^N}{V(\mathbf{u})}$$

for pairwise distinct $u_i \in \mathbb{F}$. Note that the denominator is precisely the Vandermonde determinant

$$V((u_1, \dots, u_N)) := \det(u_i^{N-j}) = \prod_{1 \leqslant i < j \leqslant N} (u_i - u_j).$$

Schur polynomials

Given a decreasing N-tuple of integers $n_1 > \cdots > n_N \geqslant 0$, the corresponding **Schur polynomial** over a field $\mathbb{F}$ (say char $\mathbb{F} = 0$) is the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{\mathbf{n}}(u_1, \ldots, u_N) := \frac{\det(u_i^{n_j})_{i,j=1}^N}{\det(u_i^{N-j})} = \frac{\det(u_i^{n_j})_{i,j=1}^N}{V(\mathbf{u})}$$

for pairwise distinct $u_i \in \mathbb{F}$. Note that the denominator is precisely the Vandermonde determinant

$$V((u_1, \ldots, u_N)) := \det(u_i^{N-j}) = \prod_{1 \leqslant i < j \leqslant N} (u_i - u_j).$$

- Basis of homogeneous symmetric polynomials in $u_1, \ldots, u_N$.
- Characters of irreducible polynomial representations of $GL_N(\mathbb{C})$, usually defined in terms of semi-standard Young tableaux.

Schur polynomials

Given a decreasing N-tuple of integers $n_1 > \cdots > n_N \geqslant 0$, the corresponding **Schur polynomial** over a field $\mathbb{F}$ (say char $\mathbb{F} = 0$) is the unique polynomial extension to $\mathbb{F}^N$ of

$$s_{\mathbf{n}}(u_1, \dots, u_N) := \frac{\det(u_i^{n_j})_{i,j=1}^N}{\det(u_i^{N-j})} = \frac{\det(u_i^{n_j})_{i,j=1}^N}{V(\mathbf{u})}$$

for pairwise distinct $u_i \in \mathbb{F}$. Note that the denominator is precisely the Vandermonde determinant

$$V((u_1, \dots, u_N)) := \det(u_i^{N-j}) = \prod_{1 \leqslant i < j \leqslant N} (u_i - u_j).$$

- Basis of homogeneous symmetric polynomials in $u_1, \dots, u_N$.
- Characters of irreducible polynomial representations of $GL_N(\mathbb{C})$, usually defined in terms of semi-standard Young tableaux.
- Weyl Character (Dimension) Formula in Type A:

$$s_{\mathbf{n}}(1, \dots, 1) = \prod_{1 \leqslant i < j \leqslant N} \frac{n_i - n_j}{j - i} = \frac{V(\mathbf{n})}{V((N-1, \dots, 1, 0))}.$$

Schur polynomials via semi-standard Young tableaux

Schur polynomials are also defined using semi-standard Young tableaux:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (4, 2, 0)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur polynomials via semi-standard Young tableaux

Schur polynomials are also defined using semi-standard Young tableaux:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (4, 2, 0)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} & s_{(4,2,0)}(u_1, u_2, u_3) \\ = \; & u_1^2 u_2 + u_1^2 u_3 + u_1 u_2^2 + 2u_1 u_2 u_3 + u_1 u_3^2 + u_2^2 u_3 + u_2 u_3^2 \\ = \; & (u_1 + u_2)(u_2 + u_3)(u_3 + u_1). \end{aligned}$$

Schur polynomials via semi-standard Young tableaux

Schur polynomials are also defined using semi-standard Young tableaux:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (4, 2, 0)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} & s_{(4,2,0)}(u_1, u_2, u_3) \\ = \ & u_1^2 u_2 + u_1^2 u_3 + u_1 u_2^2 + 2u_1 u_2 u_3 + u_1 u_3^2 + u_2^2 u_3 + u_2 u_3^2 \\ = \ & (u_1 + u_2)(u_2 + u_3)(u_3 + u_1). \end{aligned}$$

Example 2: Suppose $N = 3$ and $\mathbf{n} = (3, 2, 0)$:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$$

Then $s_{(3,2,0)}(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 + u_2 u_3$.

Cauchy's – and Frobenius's – determinantal identity

Theorem (Cauchy, 1841 memoir)

If $f(t) = (1-t)^{-1} = 1 + t + t^2 + \cdots$, *and* $f[A] := (f(a_{ij}))$, *then*

$$\det f[\mathbf{u}\mathbf{v}^T] = \det((1 - u_i v_j)^{-1})_{i,j=1}^N = \sum_{M \geqslant 0} \sum_{\mathbf{n} \vdash M} V(\mathbf{u})V(\mathbf{v}) \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}).$$

Cauchy's – and Frobenius's – determinantal identity

Theorem (Cauchy, 1841 memoir)

If $f(t) = (1-t)^{-1} = 1 + t + t^2 + \cdots$, *and* $f[A] := (f(a_{ij}))$, *then*

$$\det f[\mathbf{u}\mathbf{v}^T] = \det((1-u_i v_j)^{-1})_{i,j=1}^N = \sum_{M \geqslant 0} \sum_{\mathbf{n} \vdash M} V(\mathbf{u})V(\mathbf{v}) \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}).$$

This is the $c = 0$ special case of:

Theorem (Frobenius, *J. reine Angew. Math.* 1882)

If $f(t) = \dfrac{1-ct}{1-t}$ *for a scalar* c, *then*

$$\det f[\mathbf{u}\mathbf{v}^T] = \det \left(\frac{1 - c u_i v_j}{1 - u_i v_j} \right)_{i,j=1}^N$$

$$= V(\mathbf{u})V(\mathbf{v})(1-c)^{N-1} \left(\sum_{\mathbf{n}\,:\, n_N = 0} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) + (1-c) \sum_{\mathbf{n}\,:\, n_N > 0} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \right).$$

What happens for other power series?

The determinantal identity for polynomials

- Suppose $f(t) = f_1 t^{n_1} + \cdots + f_k t^{n_k}$ is any polynomial with $< N$ terms. (Say $n_1 > \cdots > n_k \geqslant 0$.) Then
$$f[\mathbf{u}\mathbf{v}^T] = f_1 \mathbf{u}^{\circ n_1} (\mathbf{v}^{\circ n_1})^T + \cdots + f_k \mathbf{u}^{\circ n_k} (\mathbf{v}^{\circ n_k})^T$$
has rank $k < N$, so its determinant is zero.

The determinantal identity for polynomials

- Suppose $f(t) = f_1 t^{n_1} + \cdots + f_k t^{n_k}$ is any polynomial with $< N$ terms. (Say $n_1 > \cdots > n_k \geqslant 0$.) Then
$$f[\mathbf{u}\mathbf{v}^T] = f_1 \mathbf{u}^{\circ n_1} (\mathbf{v}^{\circ n_1})^T + \cdots + f_k \mathbf{u}^{\circ n_k} (\mathbf{v}^{\circ n_k})^T$$
has rank $k < N$, so its determinant is zero.

- (Folklore case: Jacobi, Cauchy, Schur...) Suppose $f(t) = \sum_{j=1}^N f_j t^{n_j}$.

The determinantal identity for polynomials

- Suppose $f(t) = f_1 t^{n_1} + \cdots + f_k t^{n_k}$ is any polynomial with $< N$ terms. (Say $n_1 > \cdots > n_k \geqslant 0$.) Then
$$f[\mathbf{u}\mathbf{v}^T] = f_1 \mathbf{u}^{\circ n_1} (\mathbf{v}^{\circ n_1})^T + \cdots + f_k \mathbf{u}^{\circ n_k} (\mathbf{v}^{\circ n_k})^T$$
has rank $k < N$, so its determinant is zero.

- (Folklore case: Jacobi, Cauchy, Schur...) Suppose $f(t) = \sum_{j=1}^N f_j t^{n_j}$. Then $f[\mathbf{u}\mathbf{v}^T]$ factorizes as
$$\begin{pmatrix} u_1^{n_1} & u_1^{n_2} & \cdots & u_1^{n_N} \\ u_2^{n_1} & u_2^{n_2} & \cdots & u_2^{n_N} \\ \vdots & \vdots & \ddots & \vdots \\ u_N^{n_1} & u_N^{n_2} & \cdots & u_N^{n_N} \end{pmatrix} \cdot \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_N \end{pmatrix} \cdot \begin{pmatrix} v_1^{n_1} & v_1^{n_2} & \cdots & v_1^{n_N} \\ v_2^{n_1} & v_2^{n_2} & \cdots & v_2^{n_N} \\ \vdots & \vdots & \ddots & \vdots \\ v_N^{n_1} & v_N^{n_2} & \cdots & v_N^{n_N} \end{pmatrix}^T,$$
so $\det f[\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \prod_{j=1}^N f_j \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v})$.

The determinantal identity for polynomials

- Suppose $f(t) = f_1 t^{n_1} + \cdots + f_k t^{n_k}$ is any polynomial with $< N$ terms. (Say $n_1 > \cdots > n_k \geqslant 0$.) Then
$$f[\mathbf{u}\mathbf{v}^T] = f_1 \mathbf{u}^{\circ n_1} (\mathbf{v}^{\circ n_1})^T + \cdots + f_k \mathbf{u}^{\circ n_k} (\mathbf{v}^{\circ n_k})^T$$
has rank $k < N$, so its determinant is zero.

- (Folklore case: Jacobi, Cauchy, Schur...) Suppose $f(t) = \sum_{j=1}^N f_j t^{n_j}$.
Then $f[\mathbf{u}\mathbf{v}^T]$ factorizes as
$$\begin{pmatrix} u_1^{n_1} & u_1^{n_2} & \cdots & u_1^{n_N} \\ u_2^{n_1} & u_2^{n_2} & \cdots & u_2^{n_N} \\ \vdots & \vdots & \ddots & \vdots \\ u_N^{n_1} & u_N^{n_2} & \cdots & u_N^{n_N} \end{pmatrix} \cdot \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_N \end{pmatrix} \cdot \begin{pmatrix} v_1^{n_1} & v_1^{n_2} & \cdots & v_1^{n_N} \\ v_2^{n_1} & v_2^{n_2} & \cdots & v_2^{n_N} \\ \vdots & \vdots & \ddots & \vdots \\ v_N^{n_1} & v_N^{n_2} & \cdots & v_N^{n_N} \end{pmatrix}^T,$$
so $\det f[\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \prod_{j=1}^N f_j \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v})$.

- Similar computation for arbitrary polynomials – $f[\mathbf{u}\mathbf{v}^T]$ factorizes, so use the Cauchy–Binet formula.

Connection from analysis

Loewner studied $\det f[t\mathbf{u}\mathbf{u}^T]$ as a function of t (for f smooth), and computed its Taylor coefficients:

- Fix $\mathbf{u} = (u_1, \dots, u_N)^T \in \mathbb{R}^N$, with $u_i > 0$ pairwise distinct.
- Define $\Delta(t) := \det f[t\mathbf{u}\mathbf{u}^T]$, and *compute its first* $\binom{N}{2} + 1$ *derivatives*:

Connection from analysis

Loewner studied $\det f[t\mathbf{u}\mathbf{u}^T]$ as a function of t (for f smooth), and computed its Taylor coefficients:

- Fix $\mathbf{u} = (u_1, \dots, u_N)^T \in \mathbb{R}^N$, with $u_i > 0$ pairwise distinct.
- Define $\Delta(t) := \det f[t\mathbf{u}\mathbf{u}^T]$, and *compute its first* $\binom{N}{2} + 1$ *derivatives*:

$$\Delta(0) = \Delta'(0) = \cdots = \Delta^{(\binom{N}{2}-1)}(0) = 0, \quad \text{and}$$

$$\frac{\Delta^{(\binom{N}{2})}(0)}{\binom{N}{2}!} = V(\mathbf{u})^2 \cdot 1^2 \cdot \frac{f(0)}{0!} \frac{f'(0)}{1!} \cdots \frac{f^{(N-1)}(0)}{(N-1)!}.$$

Connection from analysis

Loewner studied $\det f[t\mathbf{u}\mathbf{u}^T]$ as a function of t (for f smooth), and computed its Taylor coefficients:

- Fix $\mathbf{u} = (u_1, \dots, u_N)^T \in \mathbb{R}^N$, with $u_i > 0$ pairwise distinct.
- Define $\Delta(t) := \det f[t\mathbf{u}\mathbf{u}^T]$, and *compute its first* $\binom{N}{2} + 1$ *derivatives*:

$$\Delta(0) = \Delta'(0) = \cdots = \Delta^{(\binom{N}{2}-1)}(0) = 0, \quad \text{and}$$

$$\frac{\Delta^{(\binom{N}{2})}(0)}{\binom{N}{2}!} = V(\mathbf{u})^2 \cdot 1^2 \cdot \frac{f(0)}{0!} \frac{f'(0)}{1!} \cdots \frac{f^{(N-1)}(0)}{(N-1)!}.$$

(Loewner stopped here for his purposes – of matrix positivity preservers – but)
What if Loewner had gone one step further?

Connection from analysis

Loewner studied $\det f[t\mathbf{u}\mathbf{u}^T]$ as a function of t (for f smooth), and computed its Taylor coefficients:

- Fix $\mathbf{u} = (u_1, \dots, u_N)^T \in \mathbb{R}^N$, with $u_i > 0$ pairwise distinct.
- Define $\Delta(t) := \det f[t\mathbf{u}\mathbf{u}^T]$, and *compute its first* $\binom{N}{2} + 1$ *derivatives*:

$$\Delta(0) = \Delta'(0) = \cdots = \Delta^{(\binom{N}{2}-1)}(0) = 0, \quad \text{and}$$

$$\frac{\Delta^{(\binom{N}{2})}(0)}{\binom{N}{2}!} = V(\mathbf{u})^2 \cdot 1^2 \cdot \frac{f(0)}{0!} \frac{f'(0)}{1!} \cdots \frac{f^{(N-1)}(0)}{(N-1)!}.$$

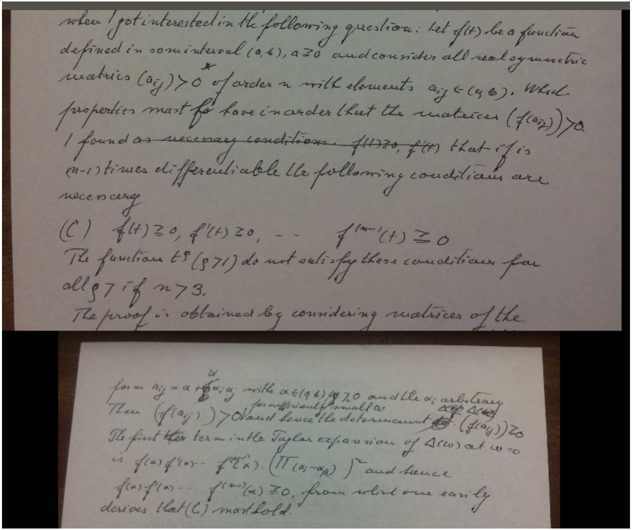
(Loewner stopped here for his purposes – of matrix positivity preservers – but) *What if Loewner had gone one step further?*

$$\frac{\Delta^{(\binom{N}{2}+1)}(0)}{(\binom{N}{2}+1)!} = V(\mathbf{u})^2 \cdot (u_1 + \cdots + u_N)^2 \cdot \frac{f(0)}{0!} \frac{f'(0)}{1!} \cdots \frac{f^{(N-2)}(0)}{(N-2)!} \cdot \frac{f^{(N)}(0)}{N!}.$$

Hidden inside this derivative is a Schur polynomial!

Loewner's calculations

Loewner had summarized these computations in a letter to Josephine Mitchell (Penn. State) on 24 Oct 1967. (Later in: Roger Horn, [*Trans. AMS* 1969].)



when I got interested in the following question: Let $f(t)$ be a function defined in some interval (a,b), $a \geq 0$ and consider all real symmetric matrices $(a_{ij}) > 0$* of order n with elements $a_{ij} \in (a,b)$. What properties must f have in order that the matrices $(f(a_{ij})) > 0$.
I found ~~as necessary conditions: $f(t) \geq 0$, $f'(t)$~~ that if f is $(n-1)$ times differentiable the following conditions are necessary

(C) $f(t) \geq 0,\ f'(t) \geq 0,\ \ldots\ f^{(n-1)}(t) \geq 0$

The functions t^ρ $(\rho > 1)$ do not satisfy these conditions for all $\rho > 1$ if $n > 3$.
The proof is obtained by considering matrices of the form $a_{ij} = \alpha + \omega \alpha_i \alpha_j$ with $\alpha \in (a,b)$, $\omega > 0$ and the α_i arbitrary. Then $(f(a_{ij})) > 0$ for sufficiently small ω and hence the determinant $\Delta(\omega) = \det (f(a_{ij})) \geq 0$. The first term in the Taylor expansion of $\Delta(\omega)$ at $\omega = 0$ is $f(\alpha) f'(\alpha) \cdots f^{(n-1)}(\alpha) \cdot \left(\prod (\alpha_i - \alpha_\beta)\right)^2$ and hence $f(\alpha) f'(\alpha) \cdots f^{(n-1)}(\alpha) \geq 0$, from which one easily derives that (C) must hold.

From each smooth function to all Schur polynomials

This provides a novel bridge, between analysis and symmetric function theory:

Given $f : [0, \epsilon) \to \mathbb{R}$ *smooth, and* $u_1, \dots, u_N > 0$ *pairwise distinct (for* $\epsilon > 0$ *and* $N \geqslant 1$*),*
set $\Delta(t) := \det f[t\mathbf{u}\mathbf{u}^T]$ *and compute* $\Delta^{(M)}(0)$ *for all integers* $M \geqslant 0$*.*

From each smooth function to all Schur polynomials

This provides a novel bridge, between analysis and symmetric function theory:

Given $f : [0, \epsilon) \to \mathbb{R}$ *smooth, and* $u_1, \dots, u_N > 0$ *pairwise distinct (for* $\epsilon > 0$ *and* $N \geqslant 1$*),*
set $\Delta(t) := \det f[t\mathbf{u}\mathbf{u}^T]$ *and compute* $\Delta^{(M)}(0)$ *for all integers* $M \geqslant 0$*.*

Uncovers *all* Schur polynomials – for $\mathbf{u}$ and $\mathbf{v}$:

Theorem (K., *Trans. Amer. Math. Soc.* 2022)

Suppose f, ϵ, N *are as above. Fix* $\mathbf{u}, \mathbf{v} \in (0, \infty)^N$ *and set* $\Delta(t) := \det f[t\mathbf{u}\mathbf{v}^T]$*. Then for all* $M \geqslant 0$*,*

$$\frac{\Delta^{(M)}(0)}{M!} = \sum_{\mathbf{n}=(n_1,\dots,n_N) \vdash M} V(\mathbf{u})V(\mathbf{v}) \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \cdot \prod_{j=1}^N \frac{f^{(n_j)}(0)}{n_j!}.$$

- All Schur polynomials "occur" inside each smooth function.

From each smooth function to all Schur polynomials

This provides a novel bridge, between analysis and symmetric function theory:

Given $f : [0,\epsilon) \to \mathbb{R}$ *smooth, and* $u_1, \dots, u_N > 0$ *pairwise distinct (for* $\epsilon > 0$ *and* $N \geqslant 1$*),*
set $\Delta(t) := \det f[t\mathbf{u}\mathbf{u}^T]$ *and compute* $\Delta^{(M)}(0)$ *for all integers* $M \geqslant 0$.

Uncovers *all* Schur polynomials – for $\mathbf{u}$ and $\mathbf{v}$:

Theorem (K., *Trans. Amer. Math. Soc.* 2022)

Suppose f, ϵ, N *are as above. Fix* $\mathbf{u}, \mathbf{v} \in (0,\infty)^N$ *and set* $\Delta(t) := \det f[t\mathbf{u}\mathbf{v}^T]$. *Then for all* $M \geqslant 0$,

$$\frac{\Delta^{(M)}(0)}{M!} = \sum_{\mathbf{n}=(n_1,\dots,n_N)\,\vdash M} V(\mathbf{u})V(\mathbf{v}) \cdot s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \cdot \prod_{j=1}^{N} \frac{f^{(n_j)}(0)}{n_j!}.$$

- All Schur polynomials "occur" inside each smooth function.
- If f is a power series, then so is Δ. What is its expansion? (Starting with Cauchy and Frobenius...)

Cauchy–Frobenius identity for all power series

Theorem (K., *Trans. Amer. Math. Soc.* 2022)

Fix a commutative unital ring R *and let* t *be an indeterminate.*
Let $f(t) := \sum_{M \geqslant 0} f_M t^M \in R[[t]]$ *be an arbitrary formal power series.*
Given vectors $\mathbf{u}, \mathbf{v} \in R^N$ *for some* $N \geqslant 1$*, we have:*

$$\det f[t\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \sum_{M \geqslant \binom{N}{2}} t^M \sum_{\mathbf{n}=(n_1,\dots,n_N) \vdash M} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \prod_{j=1}^N f_{n_j}.$$

Also true in the real-analytic topology, for $R = \mathbb{R}$ *and* $|t| <$ *radius of conv.*

Cauchy–Frobenius identity for all power series

Theorem (K., *Trans. Amer. Math. Soc.* 2022)

Fix a commutative unital ring R and let t be an indeterminate. Let $f(t) := \sum_{M \geqslant 0} f_M t^M \in R[[t]]$ be an arbitrary formal power series. Given vectors $\mathbf{u}, \mathbf{v} \in R^N$ for some $N \geqslant 1$, we have:

$$\det f[t\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \sum_{M \geqslant \binom{N}{2}} t^M \sum_{\mathbf{n}=(n_1,\dots,n_N) \vdash M} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) \prod_{j=1}^N f_{n_j}.$$

Also true in the real-analytic topology, for $R = \mathbb{R}$ and $|t| <$ radius of conv.

Similar questions and results (on symmetric function identities), including by

- Andrews–Goulden–Jackson [*Trans. Amer. Math. Soc.* 1988].
- Laksov–Lascoux–Thorup [*Acta Math.* 1989].
- Kuperberg [*Ann. of Math.* 2002].
- Ishikawa, Okado, and coauthors [*Adv. Appl. Math.* 2006, 2013].
- See also Krattenthaler, *Advanced determinantal calculus* (and its sequel) in 1998, 2005.

From determinants to all immanants

Theorem (K.–Sahi, 2022)

With (algebraic) notation as above, say over characteristic zero:

$$\operatorname{perm} f[t\mathbf{u}\mathbf{v}^T] = \frac{1}{N!} \sum_{\mathbf{m} \geqslant \mathbf{0}} t^{m_1 + \cdots + m_N} \prod_{j=1}^{N} f_{m_j} \cdot \operatorname{perm}(u_i^{m_j}) \operatorname{perm}(v_i^{m_j}).$$

Also,

From determinants to all immanants

Theorem (K.–Sahi, 2022)

With (algebraic) notation as above, say over characteristic zero:

$$\operatorname{perm} f[t\mathbf{u}\mathbf{v}^T] = \frac{1}{N!} \sum_{\mathbf{m} \geqslant \mathbf{0}} t^{m_1+\cdots+m_N} \prod_{j=1}^{N} f_{m_j} \cdot \operatorname{perm}(u_i^{m_j}) \operatorname{perm}(v_i^{m_j}).$$

Also, analogues for:

- *All irreducible characters/immanants of S_N, or of subgroups of S_N.*
- *"Fermionic" (u_i anti-commuting) analogues of these "Bosonic" results.*

Question: Fermionic/immanant versions of other symmetric function identities?

Schur polynomials in analysis: entrywise functions

- The Schur polynomials lurking inside all smooth functions (Loewner 1969 / K. 2022) turn out to play a crucial role in understanding entrywise polynomial maps that preserve positive semidefiniteness on $N \times N$ matrices.

- They are algebraic characters, but need to be studied as *functions on the positive orthant* $(0, \infty)^N$.

Schur polynomials via semi-standard Young tableaux

Back to the two examples above:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (4, 2, 0)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} & s_{(4,2,0)}(u_1, u_2, u_3) \\ = {} & u_1^2 u_2 + u_1^2 u_3 + u_1 u_2^2 + 2u_1 u_2 u_3 + u_1 u_3^2 + u_2^2 u_3 + u_2 u_3^2 \\ = {} & (u_1 + u_2)(u_2 + u_3)(u_3 + u_1). \end{aligned}$$

Example 2: Suppose $N = 3$ and $\mathbf{n} = (3, 2, 0)$:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$$

Then $s_{(3,2,0)}(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 + u_2 u_3$.

Schur polynomials via semi-standard Young tableaux

Back to the two examples above:

Example 1: Suppose $N = 3$ and $\mathbf{m} := (4, 2, 0)$. The tableaux are:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{aligned} & s_{(4,2,0)}(u_1, u_2, u_3) \\ = {} & u_1^2 u_2 + u_1^2 u_3 + u_1 u_2^2 + 2u_1 u_2 u_3 + u_1 u_3^2 + u_2^2 u_3 + u_2 u_3^2 \\ = {} & (u_1 + u_2)(u_2 + u_3)(u_3 + u_1). \end{aligned}$$

Example 2: Suppose $N = 3$ and $\mathbf{n} = (3, 2, 0)$:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$$

Then $s_{(3,2,0)}(u_1, u_2, u_3) = u_1 u_2 + u_1 u_3 + u_2 u_3$.

Note: Both polynomials are coordinate-wise non-decreasing on $(0, \infty)^N$.

Schur Monotonicity Lemma

Example: The ratio $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ for $\mathbf{m} = (4,2,0)$, $\mathbf{n} = (3,2,0)$ is:

$$f(u_1,u_2,u_3) = \frac{(u_1+u_2)(u_2+u_3)(u_3+u_1)}{u_1u_2+u_2u_3+u_3u_1}, \qquad u_1,u_2,u_3 > 0.$$

Note: both numerator and denominator are monomial-positive (in fact Schur-positive, obviously) – hence non-decreasing in each coordinate.

Fact: *Their ratio $f(\mathbf{u})$ has the same property!*

Schur Monotonicity Lemma

Example: The ratio $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ for $\mathbf{m} = (4,2,0),\ \mathbf{n} = (3,2,0)$ is:

$$f(u_1,u_2,u_3) = \frac{(u_1+u_2)(u_2+u_3)(u_3+u_1)}{u_1u_2+u_2u_3+u_3u_1}, \qquad u_1,u_2,u_3 > 0.$$

Note: both numerator and denominator are monomial-positive (in fact Schur-positive, obviously) – hence non-decreasing in each coordinate.

Fact: *Their ratio $f(\mathbf{u})$ has the same property!*

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

For integer tuples $n_1 > \cdots > n_N \geqslant 0$ and $m_1 > \cdots > m_N \geqslant 0$ such that $m_j \geqslant n_j\ \forall j$, the function

$$f : (0,\infty)^N \to \mathbb{R}, \qquad f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$$

is non-decreasing in each coordinate.
(In fact, a stronger Schur positivity phenomenon holds.)

Schur Monotonicity Lemma (cont.)

Claim: *The ratio* $f(u_1, u_2, u_3) = \dfrac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1 u_2 + u_2 u_3 + u_3 u_1}$,

treated as a ***function*** *on the orthant* $(0, \infty)^3$, *is coordinate-wise non-decreasing.*

Schur Monotonicity Lemma (cont.)

Claim: *The ratio* $f(u_1, u_2, u_3) = \dfrac{(u_1 + u_2)(u_2 + u_3)(u_3 + u_1)}{u_1 u_2 + u_2 u_3 + u_3 u_1}$,

treated as a ***function*** *on the orthant* $(0, \infty)^3$, *is coordinate-wise non-decreasing.*

(Why?) Applying the quotient rule of differentiation to f,

$$s_{\mathbf{n}}(\mathbf{u}) \partial_{u_3} s_{\mathbf{m}}(\mathbf{u}) - s_{\mathbf{m}}(\mathbf{u}) \partial_{u_3} s_{\mathbf{n}}(\mathbf{u}) = (u_1 + u_2)(u_1 u_3 + 2 u_1 u_2 + u_2 u_3) u_3,$$

and this is monomial-positive (hence numerically positive).

Schur Monotonicity Lemma (cont.)

Claim: *The ratio* $f(u_1, u_2, u_3) = \dfrac{(u_1+u_2)(u_2+u_3)(u_3+u_1)}{u_1u_2 + u_2u_3 + u_3u_1}$,

treated as a ***function*** *on the orthant* $(0,\infty)^3$, *is coordinate-wise non-decreasing.*

(Why?) Applying the quotient rule of differentiation to f,

$$s_{\mathbf{n}}(\mathbf{u})\partial_{u_3} s_{\mathbf{m}}(\mathbf{u}) - s_{\mathbf{m}}(\mathbf{u})\partial_{u_3} s_{\mathbf{n}}(\mathbf{u}) = (u_1+u_2)(u_1u_3 + 2u_1u_2 + u_2u_3)u_3,$$

and this is monomial-positive (hence numerically positive).

In fact, upon writing this as $\sum_{j\geqslant 0} p_j(u_1,u_2)u_3^j$, each p_j is Schur-positive, i.e. a sum of Schur polynomials:

$$p_0(u_1,u_2) = 0,$$

$$p_1(u_1,u_2) = 2u_1^2u_2 + 2u_1u_2^2 = 2\,\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + 2\,\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} = 2s_{(3,1)}(u_1,u_2),$$

$$p_2(u_1,u_2) = (u_1+u_2)^2 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} + \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

$$= s_{(3,0)}(u_1,u_2) + s_{(2,1)}(u_1,u_2).$$

Proof-sketch of Schur Monotonicity Lemma

The proof for general $\mathbf{m} \geqslant \mathbf{n}$ is similar:

By symmetry, and the quotient rule of differentiation, it suffices to show that

$$s_{\mathbf{n}} \cdot \partial_{u_N}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_N}(s_{\mathbf{n}})$$

is numerically positive on $(0, \infty)^N$. (Note, the coefficients in $s_{\mathbf{n}}(\mathbf{u})$ of each u_N^j are skew-Schur polynomials in $u_1, \dots, u_{N-1}$.)

Proof-sketch of Schur Monotonicity Lemma

The proof for general $\mathbf{m} \geqslant \mathbf{n}$ is similar:

By symmetry, and the quotient rule of differentiation, it suffices to show that

$$s_{\mathbf{n}} \cdot \partial_{u_N}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_N}(s_{\mathbf{n}})$$

is numerically positive on $(0, \infty)^N$. (Note, the coefficients in $s_{\mathbf{n}}(\mathbf{u})$ of each u_N^j are skew-Schur polynomials in $u_1, \dots, u_{N-1}$.)

The assertion would follow if this expression is monomial-positive.

Proof-sketch of Schur Monotonicity Lemma

The proof for general $\mathbf{m} \geqslant \mathbf{n}$ is similar:

By symmetry, and the quotient rule of differentiation, it suffices to show that

$$s_{\mathbf{n}} \cdot \partial_{u_N}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_N}(s_{\mathbf{n}})$$

is numerically positive on $(0, \infty)^N$. (Note, the coefficients in $s_{\mathbf{n}}(\mathbf{u})$ of each u_N^j are skew-Schur polynomials in $u_1, \dots, u_{N-1}$.)

The assertion would follow if this expression is monomial-positive.

Our Schur Monotonicity Lemma in fact shows that the coefficient of each u_N^j is (also) Schur-positive.

Proof-sketch of Schur Monotonicity Lemma

The proof for general $\mathbf{m} \geqslant \mathbf{n}$ is similar:

By symmetry, and the quotient rule of differentiation, it suffices to show that

$$s_{\mathbf{n}} \cdot \partial_{u_N}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_N}(s_{\mathbf{n}})$$

is numerically positive on $(0,\infty)^N$. (Note, the coefficients in $s_{\mathbf{n}}(\mathbf{u})$ of each u_N^j are skew-Schur polynomials in $u_1, \dots, u_{N-1}$.)

The assertion would follow if this expression is **monomial-positive**.

Our Schur Monotonicity Lemma in fact shows that the coefficient of each u_N^j is (also) Schur-positive.

Key ingredient: Schur-positivity result by Lam–Postnikov–Pylyavskyy (*Amer. J. Math.* 2007). □

[In turn, this emerged out of Skandera's 2004 results on determinant inequalities for totally non-negative matrices.]

Weak majorization through Schur polynomials

- Our Schur Monotonicity Lemma implies in particular:

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1,\dots,1)}{s_{\mathbf{n}}(1,\dots,1)} = \frac{V(\mathbf{m})}{V(\mathbf{n})}, \qquad \forall \mathbf{u} \in [1,\infty)^N.$$

  if $\mathbf{m}$ dominates $\mathbf{n}$ coordinate-wise.

- "Natural" to ask: for which other tuples $\mathbf{m}, \mathbf{n}$ does this inequality hold?

Weak majorization through Schur polynomials

- Our Schur Monotonicity Lemma implies in particular:

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1,\ldots,1)}{s_{\mathbf{n}}(1,\ldots,1)} = \frac{V(\mathbf{m})}{V(\mathbf{n})}, \qquad \forall \mathbf{u} \in [1,\infty)^N.$$

  if $\mathbf{m}$ dominates $\mathbf{n}$ coordinate-wise.

- "Natural" to ask: for which other tuples $\mathbf{m}, \mathbf{n}$ does this inequality hold?

Now extended to *real* tuples (generalized Vandermonde determinants):

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

Given reals $n_1 > \cdots > n_N$ *and* $m_1 > \cdots > m_N$, *TFAE:*

1. $\frac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \frac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in [1,\infty)^N_{\neq}$.
2. $\mathbf{m}$ *weakly majorizes* $\mathbf{n}$ *– i.e.,* $m_1 + \cdots + m_k \geqslant n_1 + \cdots + n_k \ \forall k$.

Weak majorization through Schur polynomials

- Our Schur Monotonicity Lemma implies in particular:

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1,\dots,1)}{s_{\mathbf{n}}(1,\dots,1)} = \frac{V(\mathbf{m})}{V(\mathbf{n})}, \qquad \forall \mathbf{u} \in [1,\infty)^N.$$

  if $\mathbf{m}$ dominates $\mathbf{n}$ coordinate-wise.

- "Natural" to ask: for which other tuples $\mathbf{m}, \mathbf{n}$ does this inequality hold?

Now extended to *real* tuples (generalized Vandermonde determinants):

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

Given reals $n_1 > \cdots > n_N$ *and* $m_1 > \cdots > m_N$, *TFAE:*

1. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in [1,\infty)^N_{\neq}$.
2. $\mathbf{m}$ *weakly majorizes* $\mathbf{n}$ *– i.e.,* $m_1 + \cdots + m_k \geqslant n_1 + \cdots + n_k \ \forall k$.

Ingredients of proof: (a) "First-order" approximation of Schur polynomials; (b) Harish-Chandra–Itzykson–Zuber integral; (c) Schur convexity result.

Cuttler–Greene–Skandera conjecture

This problem was studied originally by Skandera and others in the 2010s, for integer powers, and *on the entire positive orthant* $(0, \infty)^N$:

Cuttler–Greene–Skandera conjecture

This problem was studied originally by Skandera and others in the 2010s, for integer powers, and *on the entire positive orthant* $(0,\infty)^N$:

Theorem (Cuttler–Greene–Skandera and Sra, *Eur. J. Comb.*, 2011, 2016)

Fix integers $n_1 > \cdots > n_N \geqslant 0$ and $m_1 > \cdots > m_N \geqslant 0$. Then

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1,\ldots,1)}{s_{\mathbf{n}}(1,\ldots,1)}, \qquad \forall \mathbf{u} \in (0,\infty)^N,$$

if and only if $\mathbf{m}$ majorizes $\mathbf{n}$.

Majorization = (weak majorization) + $\left(\sum_{j=1}^N m_j = \sum_{j=1}^N n_j\right)$.

Cuttler–Greene–Skandera conjecture

This problem was studied originally by Skandera and others in the 2010s, for integer powers, and *on the entire positive orthant* $(0, \infty)^N$:

Theorem (Cuttler–Greene–Skandera and Sra, *Eur. J. Comb.*, 2011, 2016)

Fix integers $n_1 > \cdots > n_N \geqslant 0$ *and* $m_1 > \cdots > m_N \geqslant 0$. *Then*

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1, \dots, 1)}{s_{\mathbf{n}}(1, \dots, 1)}, \qquad \forall \mathbf{u} \in (0, \infty)^N,$$

if and only if $\mathbf{m}$ *majorizes* $\mathbf{n}$.

Majorization $=$ (weak majorization) $+$ $\left(\sum_{j=1}^N m_j = \sum_{j=1}^N n_j \right)$.

Questions:

1. Does this characterization extend to real powers?
2. Can one use a smaller subset than the full orthant $(0, \infty)^N$, to deduce majorization?

Cuttler–Greene–Skandera conjecture

This problem was studied originally by Skandera and others in the 2010s, for integer powers, and *on the entire positive orthant* $(0,\infty)^N$:

Theorem (Cuttler–Greene–Skandera and Sra, *Eur. J. Comb.*, 2011, 2016)

Fix integers $n_1 > \cdots > n_N \geqslant 0$ *and* $m_1 > \cdots > m_N \geqslant 0$. *Then*

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1,\dots,1)}{s_{\mathbf{n}}(1,\dots,1)}, \qquad \forall \mathbf{u} \in (0,\infty)^N,$$

if and only if $\mathbf{m}$ *majorizes* $\mathbf{n}$.

Majorization = (weak majorization) + $\left(\sum_{j=1}^N m_j = \sum_{j=1}^N n_j\right)$.

Questions:

1. Does this characterization extend to real powers?
2. Can one use a smaller subset than the full orthant $(0,\infty)^N$, to deduce majorization?

Yes, and Yes:

Majorization via Vandermonde determinants

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

Given reals $n_1 > \cdots > n_N$ *and* $m_1 > \cdots > m_N$, *TFAE:*

1. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,\infty)^N_{\neq}$.
2. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,1]^N_{\neq} \cup [1,\infty)^N_{\neq}$.
3. $\mathbf{m}$ *majorizes* $\mathbf{n}$.

Majorization via Vandermonde determinants

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

Given reals $n_1 > \cdots > n_N$ *and* $m_1 > \cdots > m_N$, *TFAE:*

1. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,\infty)^N_{\neq}$.
2. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,1]^N_{\neq} \cup [1,\infty)^N_{\neq}$.
3. $\mathbf{m}$ *majorizes* $\mathbf{n}$.

Proof:

- $(1) \implies (2)$: Obvious. $(3) \implies (1)$: Akin to Sra (2016).
- $(2) \implies (3)$: If $\mathbf{u} \in [1,\infty)^N_{\neq}$, then by preceding result: $\mathbf{m} \succ_w \mathbf{n}$.

Majorization via Vandermonde determinants

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

Given reals $n_1 > \cdots > n_N$ *and* $m_1 > \cdots > m_N$, *TFAE:*

1. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,\infty)^N_{\neq}$.
2. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,1]^N_{\neq} \cup [1,\infty)^N_{\neq}$.
3. $\mathbf{m}$ *majorizes* $\mathbf{n}$.

Proof:

- $(1) \implies (2)$: Obvious. $(3) \implies (1)$: Akin to Sra (2016).
- $(2) \implies (3)$: If $\mathbf{u} \in [1,\infty)^N_{\neq}$, then by preceding result: $\mathbf{m} \succ_w \mathbf{n}$.
  If $\mathbf{u} \in (0,1]^N_{\neq}$, let $v_i := 1/u_i \geqslant 1$. Now compute:

$$\frac{\det(\mathbf{v}^{\circ(-\mathbf{m})})}{\det(\mathbf{v}^{\circ(-\mathbf{n})})} = \frac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \frac{V(\mathbf{m})}{V(\mathbf{n})} = \frac{V(-\mathbf{m})}{V(-\mathbf{n})}.$$

  By preceding result: $-\mathbf{m} \succ_w -\mathbf{n}$;

Majorization via Vandermonde determinants

Theorem (K.–Tao, *Amer. J. Math.*, 2021)

Given reals $n_1 > \cdots > n_N$ *and* $m_1 > \cdots > m_N$, *TFAE:*

1. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,\infty)^N_{\neq}$.
2. $\dfrac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \dfrac{V(\mathbf{m})}{V(\mathbf{n})}$, *for all "distinct" tuples* $\mathbf{u} \in (0,1]^N_{\neq} \cup [1,\infty)^N_{\neq}$.
3. $\mathbf{m}$ *majorizes* $\mathbf{n}$.

Proof:

- $(1) \implies (2)$: Obvious. $(3) \implies (1)$: Akin to Sra (2016).
- $(2) \implies (3)$: If $\mathbf{u} \in [1,\infty)^N_{\neq}$, then by preceding result: $\mathbf{m} \succ_w \mathbf{n}$.
  If $\mathbf{u} \in (0,1]^N_{\neq}$, let $v_i := 1/u_i \geqslant 1$. Now compute:

$$\frac{\det(\mathbf{v}^{\circ(-\mathbf{m})})}{\det(\mathbf{v}^{\circ(-\mathbf{n})})} = \frac{\det(\mathbf{u}^{\circ \mathbf{m}})}{\det(\mathbf{u}^{\circ \mathbf{n}})} \geqslant \frac{V(\mathbf{m})}{V(\mathbf{n})} = \frac{V(-\mathbf{m})}{V(-\mathbf{n})}.$$

  By preceding result: $-\mathbf{m} \succ_w -\mathbf{n}$; and $\mathbf{m} \succ_w \mathbf{n} \iff \mathbf{m}$ majorizes $\mathbf{n}$.

□

Precursors to Cuttler-Greene-Skandera (and Sra, ...)

Instead of using Schur polynomials, what if one uses other symmetric functions?

C-G-S: $\frac{s_{\mathbf{m}}(u_1, \ldots, u_N)}{s_{\mathbf{m}}(1, \ldots, 1)} \geqslant \frac{s_{\mathbf{n}}(u_1, \ldots, u_N)}{s_{\mathbf{n}}(1, \ldots, 1)}$ on $(0, \infty)^N \quad \Longleftrightarrow \quad \mathbf{m}$ majorizes $\mathbf{n}$.

Precursors to Cuttler-Greene-Skandera (and Sra, ...)

Instead of using Schur polynomials, what if one uses other symmetric functions?

C-G-S: $\dfrac{s_{\mathbf{m}}(u_1,\dots,u_N)}{s_{\mathbf{m}}(1,\dots,1)} \geqslant \dfrac{s_{\mathbf{n}}(u_1,\dots,u_N)}{s_{\mathbf{n}}(1,\dots,1)}$ on $(0,\infty)^N \iff \mathbf{m}$ majorizes $\mathbf{n}$.

Instead, if one uses the *monomial symmetric polynomial*

$$m_\lambda(u_1,\dots,u_N) := \frac{|S_N \cdot \lambda|}{N!} \sum_{\sigma \in S_N} \prod_{j=1}^{N} u_j^{\lambda_{\sigma(j)}},$$

then:

Theorem (Muirhead, *Proc. Edinburgh Math. Soc.* 1903)

Fix scalars $n_1 > \cdots > n_N \geqslant 0$ *and* $m_1 > \cdots > m_N \geqslant 0$. *Then*

$$\frac{m_{\mathbf{m}}(\mathbf{u})}{m_{\mathbf{m}}(1,\dots,1)} \geqslant \frac{m_{\mathbf{n}}(\mathbf{u})}{m_{\mathbf{n}}(1,\dots,1)}, \qquad \forall \mathbf{u} \in (0,\infty)^N$$

if and only if $\mathbf{m}$ *majorizes* $\mathbf{n}$.

Precursors to Cuttler-Greene-Skandera (and Sra, . . .)

Instead of using Schur polynomials, what if one uses other symmetric functions?

C-G-S: $\dfrac{s_{\mathbf{m}}(u_1, \dots, u_N)}{s_{\mathbf{m}}(1, \dots, 1)} \geqslant \dfrac{s_{\mathbf{n}}(u_1, \dots, u_N)}{s_{\mathbf{n}}(1, \dots, 1)}$ on $(0, \infty)^N \iff$ $\mathbf{m}$ majorizes $\mathbf{n}$.

Instead, if one uses the *monomial symmetric polynomial*

$$m_\lambda(u_1, \dots, u_N) := \frac{|S_N \cdot \lambda|}{N!} \sum_{\sigma \in S_N} \prod_{j=1}^{N} u_j^{\lambda_{\sigma(j)}},$$

then:

Theorem (Muirhead, *Proc. Edinburgh Math. Soc.* 1903)

Fix scalars $n_1 > \cdots > n_N \geqslant 0$ and $m_1 > \cdots > m_N \geqslant 0$. Then

$$\frac{m_{\mathbf{m}}(\mathbf{u})}{m_{\mathbf{m}}(1, \dots, 1)} \geqslant \frac{m_{\mathbf{n}}(\mathbf{u})}{m_{\mathbf{n}}(1, \dots, 1)}, \qquad \forall \mathbf{u} \in (0, \infty)^N$$

if and only if $\mathbf{m}$ majorizes $\mathbf{n}$.

Question: What if one restricts to $\mathbf{u} \in [1, \infty)^N$?

Majorization inequalities

The C-G-S–Sra inequality (and its follow-up by K.–Tao)
as well as Muirhead's inequality, are examples of *majorization inequalities*.

Other majorization inequalities have been shown by:

- Maclaurin (1729)
- Newton (1732)
- Schlömilch (1858)
- Schur (1920s?)
- Popoviciu (1934)
- Gantmacher (1959)

Majorization inequalities

The C-G-S–Sra inequality (and its follow-up by K.–Tao) as well as Muirhead's inequality, are examples of *majorization inequalities*.

Other majorization inequalities have been shown by:

- Maclaurin (1729)
- Newton (1732)
- Schlömilch (1858)
- Schur (1920s?)
- Popoviciu (1934)
- Gantmacher (1959)

Vast generalization by McSwiggen–Novak [*IMRN* 2022] to all Weyl groups, via spherical functions on Riemannian symmetric spaces.

Conjectured to hold even more generally, for Heckman–Opdam hypergeometric functions – this would extend C-G-S–Sra from Schur polynomials to Jack polynomials. (Extends to Macdonald polynomials?)

Complete homogeneous symmetric (CHS) polynomials

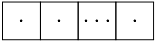
Define $h_k(u_1, u_2, \dots) := \displaystyle\sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$.

·	·	···	·

Thus, $\quad h_0 = 1, \qquad h_2 = \sum_i u_i^2 + \sum_{i<j} u_i u_j = \frac{1}{2}(h_1(\mathbf{u})^2 + p_2(\mathbf{u})) \geqslant 0.$

Complete homogeneous symmetric (CHS) polynomials

Define $h_k(u_1, u_2, \dots) := \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$.



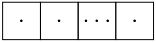
Thus, $h_0 = 1$, $h_2 = \sum_i u_i^2 + \sum_{i<j} u_i u_j = \frac{1}{2}(h_1(\mathbf{u})^2 + p_2(\mathbf{u})) \geqslant 0.$

Lemma

For all integers $r \geqslant 0$ and $N > 0$, the polynomial $h_{2r}(u_1, \dots u_N)$ does not vanish on $\mathbb{R}^N$ except at the origin.

Complete homogeneous symmetric (CHS) polynomials

Define $h_k(u_1, u_2, \dots) := \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} u_{i_1} u_{i_2} \cdots u_{i_k}.$



Thus, $h_0 = 1, \qquad h_2 = \sum_i u_i^2 + \sum_{i<j} u_i u_j = \frac{1}{2}(h_1(\mathbf{u})^2 + p_2(\mathbf{u})) \geqslant 0.$

Lemma

For all integers $r \geqslant 0$ and $N > 0$, the polynomial $h_{2r}(u_1, \dots u_N)$ does not vanish on $\mathbb{R}^N$ except at the origin.

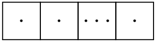
Proof (A. Barvinok): Given i.i.d. exponential(1) random variables $Z_1, \dots, Z_N$,

$$k!\, h_k(u_1, \dots, u_N) = \mathbb{E}\left[(u_1 Z_1 + \cdots + u_N Z_N)^k\right] \quad \forall k \geqslant 0,\ u_1, \dots, u_N \in \mathbb{R}.$$

□

Complete homogeneous symmetric (CHS) polynomials

Define $h_k(u_1, u_2, \dots) := \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$.



Thus, $h_0 = 1$, $h_2 = \sum_i u_i^2 + \sum_{i<j} u_i u_j = \frac{1}{2}(h_1(\mathbf{u})^2 + p_2(\mathbf{u})) \geqslant 0$.

Lemma

For all integers $r \geqslant 0$ and $N > 0$, the polynomial $h_{2r}(u_1, \dots u_N)$ does not vanish on $\mathbb{R}^N$ except at the origin.

Proof (A. Barvinok): Given i.i.d. exponential(1) random variables $Z_1, \dots, Z_N$,

$$k!\, h_k(u_1, \dots, u_N) = \mathbb{E}\left[(u_1 Z_1 + \cdots + u_N Z_N)^k\right] \quad \forall k \geqslant 0,\ u_1, \dots, u_N \in \mathbb{R}.$$

□

Which (other) Schur polynomials share this property?

CHS polynomials (cont.)

Which other Schur polynomials share this property?

CHS polynomials (cont.)

Which other Schur polynomials share this property? None:

Lemma (K.–Tao)

Suppose $N \geqslant 1$ and $n_1 > \cdots > n_N \geqslant 0$ are integers. Then the Schur polynomial $s_{\mathbf{n}}(u_1, \dots, u_N)$ is nonvanishing on $\mathbb{R}^N \setminus \{\mathbf{0}\}$, if and only if there exists $r \geqslant 0$ such that

$$n_N = 0, \quad n_{N-1} = 1, \quad \cdots, \quad n_2 = N-2, \quad n_1 = (N-1) + 2r.$$

CHS polynomials (cont.)

Which other Schur polynomials share this property? None:

Lemma (K.–Tao)

Suppose $N \geqslant 1$ and $n_1 > \cdots > n_N \geqslant 0$ are integers. Then the Schur polynomial $s_{\mathbf{n}}(u_1, \ldots, u_N)$ is nonvanishing on $\mathbb{R}^N \setminus \{\mathbf{0}\}$, if and only if there exists $r \geqslant 0$ such that

$$n_N = 0, \quad n_{N-1} = 1, \quad \cdots, \quad n_2 = N-2, \quad n_1 = (N-1) + 2r.$$

Now recall the Schur Monotonicity Lemma: if $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise, then

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} : (0, \infty)^N \to \mathbb{R}$$

is non-decreasing in each coordinate. In particular, it attains its supremum on $[0,1]^N \setminus \{\mathbf{0}\}$ at $(1, \ldots, 1)$.

CHS polynomials (cont.)

Which other Schur polynomials share this property? None:

Lemma (K.–Tao)

Suppose $N \geqslant 1$ and $n_1 > \cdots > n_N \geqslant 0$ are integers. Then the Schur polynomial $s_{\mathbf{n}}(u_1, \dots, u_N)$ is nonvanishing on $\mathbb{R}^N \setminus \{\mathbf{0}\}$, if and only if there exists $r \geqslant 0$ such that

$$n_N = 0, \quad n_{N-1} = 1, \quad \cdots, \quad n_2 = N-2, \quad n_1 = (N-1) + 2r.$$

Now recall the Schur Monotonicity Lemma: if $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise, then

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} : (0, \infty)^N \to \mathbb{R}$$

is non-decreasing in each coordinate. In particular, it attains its supremum on $[0,1]^N \setminus \{\mathbf{0}\}$ at $(1, \dots, 1)$.

Now consider the *two-sided* optimization problem, i.e. on $[-1,1]^N \setminus \{0\}$. The above Lemmas suggest taking $\mathbf{n} = (N-1+2r, N-2, \dots, 1, 0)$.

Maximizing ratios involving CHS polynomials

"Two-sided" variant: Suppose $\mathbf{n} = (N-1+2r, N-2, \dots, 1, 0))$ for $r \geqslant 0$, and $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise. Define

$$f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})^2}{h_{2r}(\mathbf{u})^2}.$$

Maximizing ratios involving CHS polynomials

"Two-sided" variant: Suppose $\mathbf{n} = (N-1+2r, N-2, \ldots, 1, 0))$ for $r \geqslant 0$, and $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise. Define

$$f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})^2}{h_{2r}(\mathbf{u})^2}.$$

Question: How does this function behave on $[-1,1]^N \setminus \{\mathbf{0}\}$? Where does it attain its supremum?

Maximizing ratios involving CHS polynomials

"Two-sided" variant: Suppose $\mathbf{n} = (N-1+2r, N-2, \dots, 1, 0))$ for $r \geqslant 0$, and $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise. Define

$$f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})^2}{h_{2r}(\mathbf{u})^2}.$$

Question: How does this function behave on $[-1,1]^N \setminus \{\mathbf{0}\}$? Where does it attain its supremum?

- By homogeneity considerations, enough to consider the behavior on the boundary of the cube $[-1,1]^N$ (a compact set). Where is the maximum attained – and what does it equal?

Maximizing ratios involving CHS polynomials

“Two-sided” variant: Suppose $\mathbf{n} = (N-1+2r, N-2, \ldots, 1, 0))$ for $r \geqslant 0$, and $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise. Define

$$f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})^2}{h_{2r}(\mathbf{u})^2}.$$

Question: How does this function behave on $[-1,1]^N \setminus \{\mathbf{0}\}$? Where does it attain its supremum?

- By homogeneity considerations, enough to consider the behavior on the boundary of the cube $[-1,1]^N$ (a compact set). Where is the maximum attained – and what does it equal?
- A solution to this question has consequences for entrywise polynomials that preserve positivity on matrices.

References I: Symmetric function identities

[1] A.-L. Cauchy. *Memoir*, **1841**.
Mémoire sur les fonctions alternées et sur les sommes alternées.

[2] F.G. Frobenius. *J. reine Angew. Math.*, **1882**.
Über die elliptischen Funktionen zweiter Art.

[3] C. Loewner. *Letter to J. Mitchell*, **1967**.
(Published+attributed in paper by R.A. Horn, in *Trans. Amer. Math. Soc.*, 1969.)

[4] G.E. Andrews, I.P. Goulden, and D.M. Jackson. *Trans. Amer. Math. Soc.*, **1988**.
Generalizations of Cauchy's summation theorem for Schur functions.

[5] D. Laksov, A. Lascoux, and A. Thorup. *Acta Math.*, **1989**.
On Giambelli's theorem for complete correlations.

[6] G. Kuperberg. *Ann. of Math.*, **2002**.
Symmetry classes of alternating-sign matrices under one roof.

[7] H. Rosengren and M. Schlosser. *Compos. Math.*, **2006**.
Elliptic determinant evaluations and Macdonald identities for affine root systems.

[8] A. Khare. *Trans. Amer. Math. Soc.*, **2022**.
Smooth entrywise positivity preservers, a Horn–Loewner master theorem, and symmetric function identities.

[9] A. Khare and S. Sahi. *Preprint*, **2022**.
From Cauchy's determinant formula to bosonic/fermionic immanant identities.

References II: Majorization inequalities

[1] C. Maclaurin. *Philos. Trans.*, **1729**.
A second letter to Martin Foulkes, Esq., concerning the roots of equations with the demonstrations of other rules in algebra.

[2] I. Newton. *Memoir*, **1732**.
Arithmetica universalis: sive de compositione et resolutione arithmetica liber.

[3] O. Schlömilch. *Z. Math. Phys.*, **1858**.
Über Mittel grössen verschiedener Ordnung.

[4] R.F. Muirhead. *Proc. Edinburgh Math. Soc.*, **1903**.
Some methods applicable to identities of symmetric algebraic functions of n letters.

[5] A. Cuttler, C. Greene, and M. Skandera. *Eur. J. Combin.*, **2011**.
Inequalities for symmetric means.

[6] S. Sra. *Eur. J. Combin.*, **2016**.
On inequalities for normalized Schur functions.

[7] A. Khare and T. Tao. *Amer. J. Math.*, **2021**.
On the sign patterns of entrywise positivity preservers in fixed dimension.

[8] C. McSwiggen and J. Novak. *Int. Res. Math. Not. IMRN*, **2022**.
Majorization and spherical functions.

Thank you for your attention.

American Institute of Mathematics

NSF

Department of Science & Technology, Government of India

सत्यमेव जयते

SERB INDIA

INDIAN INSTITUTE OF SCIENCE

भारतीय विज्ञान संस्थान

DARPA

ICMS International Centre for Mathematical Sciences

I L A S

International Linear Algebra Society

Taylor & Francis

TATA TRUSTS

Infosys Foundation

W-majorization

Let V = Euclidean space containing Φ = crystallographic root system, with Weyl group $W \subset O(V)$.
(So W is generated by the reflections in the hyperplanes orthogonal to $\alpha \in \Phi$.)

W-majorization

Let V = Euclidean space containing Φ = crystallographic root system, with Weyl group $W \subset O(V)$.
(So W is generated by the reflections in the hyperplanes orthogonal to $\alpha \in \Phi$.)

Definition *(McSwiggen–Novak):* Given $\lambda, \mu \in V$, say that λ W-**majorizes** μ if μ lies in the convex hull of the orbit $W \cdot \lambda$.

Special case: If Φ is of type A, then $W = S_N$, and then

λ S_N-majorizes μ precisely means λ majorizes μ.

Riemannian symmetric spaces

- Let $G =$ connected Lie group, $\sigma : G \to G$ an order-2 automorphism. If $K := G^\sigma$ is compact, $X = G/K$ is a *Riemannian symmetric space.*

Riemannian symmetric spaces

- Let G = connected Lie group, $\sigma : G \to G$ an order-2 automorphism. If $K := G^\sigma$ is compact, $X = G/K$ is a *Riemannian symmetric space.*
- (Under further assumptions:) Iwasawa decomposition $G = NAK$. The weights/roots of $\mathrm{Lie}(G)$ w.r.t. $\mathfrak{a} := \mathrm{Lie}(A)$ form a root system Φ.
- Now study W-majorization for $\lambda, \mu \in \mathfrak{a}$.
- The analogues of (normalized) Schur polyomials are *spherical functions*, studied by Harish-Chandra [*Amer. J. Math.* 1958].

Riemannian symmetric spaces

- Let G = connected Lie group, $\sigma : G \to G$ an order-2 automorphism. If $K := G^\sigma$ is compact, $X = G/K$ is a *Riemannian symmetric space.*
- (Under further assumptions:) Iwasawa decomposition $G = NAK$. The weights/roots of $\mathrm{Lie}(G)$ w.r.t. $\mathfrak{a} := \mathrm{Lie}(A)$ form a root system Φ.
- Now study W-majorization for $\lambda, \mu \in \mathfrak{a}$.
- The analogues of (normalized) Schur polyomials are *spherical functions*, studied by Harish-Chandra [*Amer. J. Math.* 1958].

Theorem (McSwiggen–Novak, *IMRN* 2022)

Extended the C-G-S / Sra / K.–Tao results, to characterize W-majorization on $\mathfrak{a}$, via inequalities of the spherical functions $\phi_{i\lambda} \geqslant \phi_{i\mu}$ on G/K.