Multi-grounded partitions and character formulas

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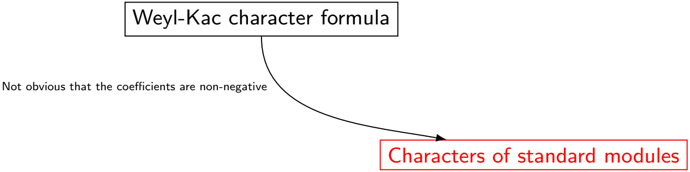

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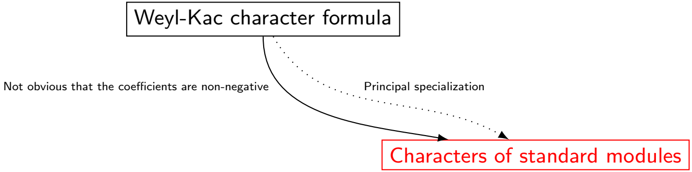

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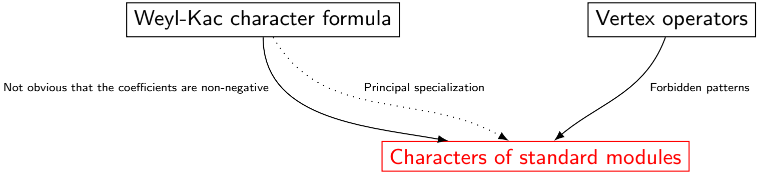

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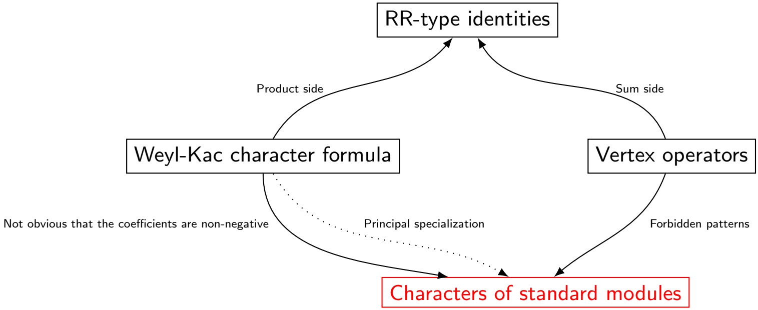

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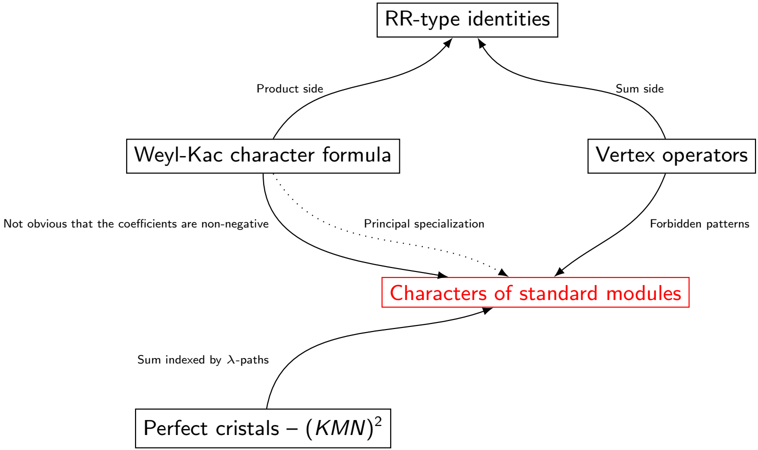

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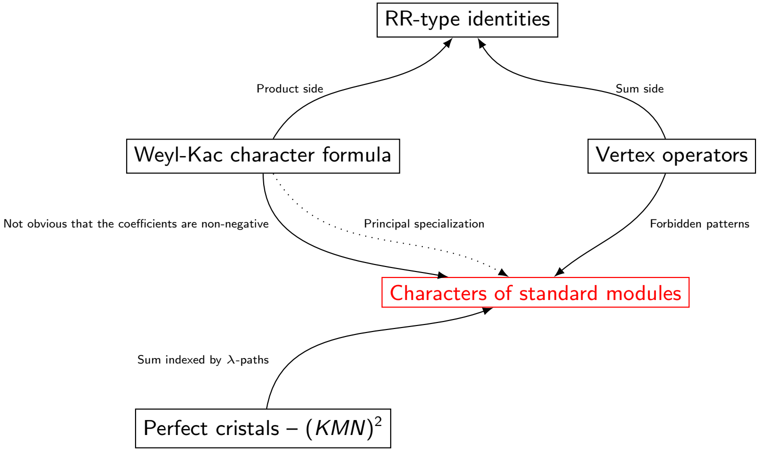

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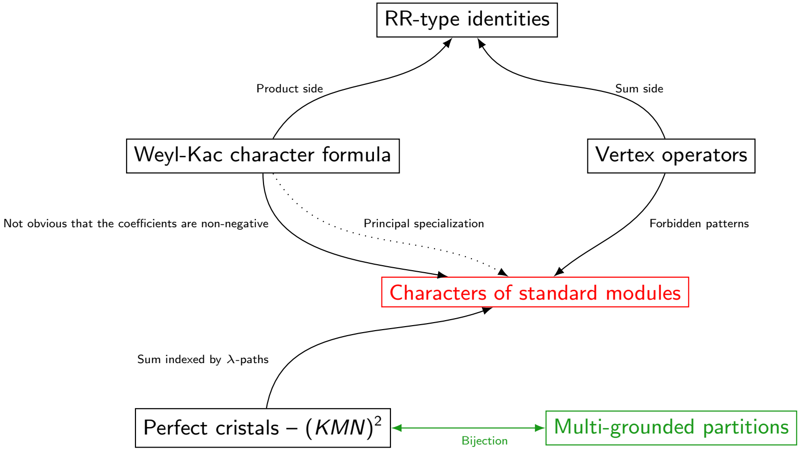

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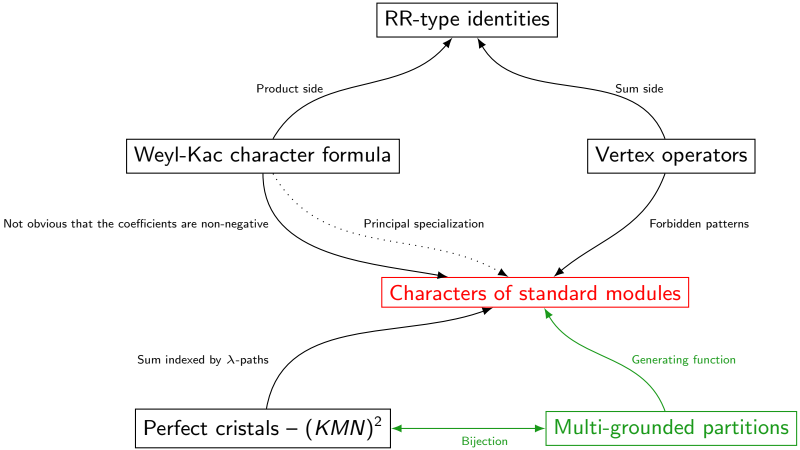

partitions

A partition is a non-increasing finite sequence of positive integers .

Example: $(4, 3, 1, 1), (1, 1, 1, 1, 1)$.

Generalized colored partitions

Let $\mathcal{C}$ be a set. Suppose that integers occur in "colors" in $\mathcal{C}$. The set of colored integers is $\mathbb{Z}_{\mathcal{C}}$. Let $\succ$ be a binary relation on $\mathbb{Z}_{\mathcal{C}}$.

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Example: $\mathcal{C} = \{c_1, c_2\}$, and let $\gg$ be the **order** defined on $\mathbb{Z}_{\mathcal{C}}$ such that

$$\cdots \succ 1_{c_2} \succ 1_{c_2} \succ 1_{c_1} \succ 1_{c_1} \succ 0_{c_2} \succ 0_{c_2} \succ 0_{c_1} \succ 0_{c_1} \succ (-1)_{c_2} \succ \cdots .$$

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A generalized colored partition according to the relation $\succ$ is a well-ordered finite sequence of colored integers according to the relation $\succ$.

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The sequence $(3_{c_1}, 3_{c_1}, 2_{c_2}, 2_{c_1})$ is allowed, but not $(2_{c_1}, 2_{c_2})$.

Multi-grounded partitions

Let $\mathcal{C}$, $\mathbb{Z}_{\mathcal{C}}$, and $\succ$ be respectively a set of colors, the set of integers colored with colors in $\mathcal{C}$, and a binary relation defined on $\mathbb{Z}_{\mathcal{C}}$. Suppose that there exist some colors $c_{g_0}, \ldots, c_{g_{t-1}}$ in $\mathcal{C}$ and **unique** colored integers $u^{(0)}_{c_{g_0}}, \ldots, u^{(t-1)}_{c_{g_{t-1}}}$ such that

$$u^{(0)} + \cdots + u^{(t-1)} = 0$$

$$u^{(0)}_{c_{g_0}} \succ u^{(1)}_{c_{g_1}} \succ \cdots \succ u^{(t-1)}_{c_{g_{t-1}}} \succ u^{(0)}_{c_{g_0}}.$$

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$$u^{(0)}_{c_{g_0}} \succ u^{(1)}_{c_{g_1}} \succ \cdots \succ u^{(t-1)}_{c_{g_{t-1}}} \succ u^{(0)}_{c_{g_0}}.$$

Then a **multi-grounded partition** with ground $c_{g_0}, \dots, c_{g_{t-1}}$ and relation $\succ$ is a non-empty generalized colored partition $\pi = (\pi_0, \dots, \pi_{s-1}, u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}})$ with relation $\succ$, such that $(\pi_{s-t}, \dots, \pi_{s-1}) \neq (u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}})$ in terms of colored integers.

Example of multi-grounded partitions

Consider the set of colors $\mathcal{C} = \{c_1, c_2, c_3\}$, the matrix

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ -2 & 0 & 2 \end{pmatrix},$$

and define the relation $\succ$ on $\mathbb{Z}_{\mathcal{C}}$ by $k_{c_b} \succ k'_{c_{b'}}$ if and only if $k - k' \geq M_{b,b'}$. If we choose $(g_0, g_1) = (1, 3)$, then $(u^{(0)}, u^{(1)}) = (1, -1)$.

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Hence, $(3_{c_3}, 3_{c_2}, 3_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ and $(1_{c_3}, 3_{c_1}, 1_{c_3}, 3_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ are examples of multi-grounded partitions with ground c_1, c_3 and relation $\succ$, while $(1_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ and $(2_{c_1}, 1_{c_1}, -1_{c_3})$ are not.

Perfect crystals

Let $\mathfrak{g}$ be an affine Kac–Moody algebra with simple positive roots $\alpha_0, \ldots, \alpha_n$ and with null root $\delta = d_0\alpha_0 + \cdots + d_n\alpha_n$. For an integer level $\ell \geq 1$ and a dominant weight λ of level ℓ, Kashiwara et al. define the notion of a *perfect crystal $\mathcal{B}$ of level ℓ*, an *energy function H*: $\mathcal{B} \otimes \mathcal{B} \to \mathbb{Z}$, and a particular element

$$\mathfrak{p}_\lambda = (g_k)_{k=0}^{\infty} = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0 \in \mathcal{B}^\infty,$$

called the *ground state path of weight λ*. From this they consider all elements of the form

$$\mathfrak{p} = (p_k)_{k=0}^{\infty} = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0 \in \mathcal{B}^\infty,$$

which satisfy $p_k = g_k$ for large enough k. Such elements are called λ-paths.

The $(KMN)^2$ character formula

Theorem ($(\mathrm{KMN})^2$ crystal base character formula)

Let λ be a dominant weight of level ℓ, let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$, and let $\mathfrak{p} = (p_k)_{k=0}^{\infty}$ be a λ-path. Then the weight of $\mathfrak{p}$ and the character of the irreducible highest weight $U_q(\widehat{\mathfrak{g}})$-module $L(\lambda)$ are given by the following expressions:

$$\mathrm{wt}\mathfrak{p} = \lambda + \sum_{k=0}^{\infty} \left((\overline{\mathrm{wt}} p_k - \overline{\mathrm{wt}} g_k) - \frac{\delta}{d_0} \sum_{j=k}^{\infty} (H(p_{j+1} \otimes p_j) - H(g_{j+1} \otimes g_j)) \right),$$

$$\mathrm{ch}(L(\lambda)) = \sum_{\mathfrak{p} \in \mathcal{P}(\lambda)} e^{\mathrm{wt}\mathfrak{p}},$$

where $\overline{\mathrm{wt}} b$ stands for the weight of the element b in $\mathcal{B}$.

Normalizing the energy function

Let $\mathcal{B}$ be a perfect crystal of level ℓ, and let λ be a level ℓ dominant weight with ground state path $\mathfrak{p}_\lambda = (g_k)_{k \geq 0}$ with period t. Let H be an energy function on $\mathcal{B} \otimes \mathcal{B}$.

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Define the function H_λ, for all $b, b' \in \mathcal{B}$, by

$$H_\lambda(b\otimes b') := H(b\otimes b') - \frac{1}{t}\sum_{k=0}^{t-1} H(g_{k+1}\otimes g_k)\,.$$

In the following, we choose a suitable divisor D of $2t$ such that $DH_\lambda(\mathcal{B}\otimes\mathcal{B}) \subset \mathbb{Z}$ and $\frac{1}{t}\sum_{k=0}^{t-1}(k+1)DH_\lambda(g_{k+1}\otimes g_k) \in \mathbb{Z}$.

Multi-grounded partition related to the energy function

Let us now consider the set of colors $\mathcal{C}_\mathcal{B}$ indexed by $\mathcal{B}$, and let us define the relation $\gg$ on $\mathbb{Z}_{\mathcal{C}_\mathcal{B}}$ by

$$k_{c_b} \gg k'_{c_{b'}} \iff k - k' \geq DH_\lambda(b' \otimes b).$$

Proposition

The set of multi-grounded partitions with ground $c_{g_0}, \dots, c_{g_{t-1}}$ and relation $\gg$ is the set of non-empty generalized colored partitions $\pi = (\pi_0, \dots, \pi_{s-1}, u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}})$ with relation $\gg$ such that $(\pi_{s-t}, \dots, \pi_{s-1}) \neq (u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}})$, and for all $k \in \{0, \dots, t-1\}$,

$$u^{(k)} = -\frac{1}{t}\sum_{j=0}^{t-1}(j+1)DH_\lambda(g_{j+1} \otimes g_j) + \sum_{j=k}^{t-1} DH_\lambda(g_{j+1} \otimes g_j).$$

Main result

Let d be a positive integer. Let $\mathcal{P}_d$ be the set of multi-grounded partitions with ground $c_{g_0}, \ldots, c_{g_{t-1}}$ and relation $\gg$ satisfying the following conditions:

- the number of parts is a multiple of t,
- the difference between two consecutive parts is a multiple of d.

Main result

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- the difference between two consecutive parts is a multiple of d.

Theorem (Dousse, K.)

Setting $q = e^{-\delta/(d_0 D)}$ *and* $c_b = e^{\overline{\mathrm{wt}} b}$ *for all* $b \in \mathcal{B}$*, we have* $c_{g_0} \cdots c_{g_{t-1}} = 1$*, and the character of the irreducible highest weight* $U_q(\mathfrak{g})$*-module* $L(\lambda)$ *is given by the following expressions:*

$$\sum_{\pi \in \mathcal{P}_d} C(\pi) q^{|\pi|} = \frac{e^{-\lambda}\mathrm{ch}(L(\lambda))}{(q^d; q^d)_\infty}.$$

Here, $C(\pi) = c_{b_0} \dots c_{b_s}$ *and* $|\pi| = k_0 + \cdots + k_s$ *for the generalized colored partition* $\pi = ((k_0)_{c_{b_0}}, \dots, (k_s)_{c_{b_s}})$.

Character for standard level 1 modules of the Lie algebra $A_{2n-1}^{(2)} (n \geq 3)$

Theorem (Dousse, K.)

Let $n \geq 3$, and let $\Lambda_0, \ldots, \Lambda_n$ be the fundamental weights and $\alpha_0, \ldots, \alpha_n$ be the simple roots of $A_{2n-1}^{(2)}$. Let $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 \cdots + 2\alpha_{n-1} + \alpha_n$ be the null root. Set

$$q = e^{-\delta/2} \quad \text{and} \quad c_i = e^{\alpha_i + \cdots + \alpha_{n-1} + \alpha_n/2} \text{ for all } i \in \{1, \ldots, n\}.$$

The two dominant weights of level 1 are Λ_0 and Λ_1, and we have

$$e^{-\Lambda_0} \mathrm{ch}(L(\Lambda_0)) = \mathcal{E}\left((q^2; q^4)_\infty \prod_{k=1}^{n} (-c_k q; q^2)_\infty (-c_k^{-1} q; q^2)_\infty \right),$$

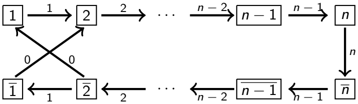
$$e^{-\Lambda_1} \mathrm{ch}(L(\Lambda_1)) = \mathcal{E}\left((q^2; q^4)_\infty (-c_1 q^3; q^2)_\infty (-c_1^{-1} q^{-1}; q^2)_\infty \prod_{k=2}^{n} (-c_k q; q^2)_\infty (-c_k^{-1} q; q^2)_\infty \right),$$

where

$$\mathcal{E}(F(c_1, \ldots, c_n)) = \frac{1}{2}(F(c_1, \ldots, c_n) + F(-c_1, \ldots, -c_n)).$$

Crystal graph $\mathcal{B}$ of the vector representation for the Lie algebra $A_{2n-1}^{(2)}(n \geq 3)$

$\mathcal{B}$:



$b^{\Lambda_0} = b_{\Lambda_1} = 1 \qquad b^{\Lambda_1} = b_{\Lambda_0} = \overline{1}$

$\mathfrak{p}_{\Lambda_0} = (\cdots \overline{1}\,1\,\overline{1}\,1\,\overline{1}) \qquad \mathfrak{p}_{\Lambda_1} = (\cdots 1\,\overline{1}\,1\,\overline{1}\,1)$

Energy function

The energy function such that $H(1 \otimes \overline{1}) = -1$, where $H(b_1 \otimes b_2)$ is the entry in column b_1 and row b_2:

$$
H = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \\ \overline{n} \\ \vdots \\ \overline{2} \\ \overline{1} \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & \cdots & n & \overline{n} & \cdots & \overline{2} & \overline{1} \end{array} \\
\left(\begin{array}{cccccccc}
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & \ddots & & & & & & \vdots \\
\vdots & \ddots & \ddots & & & 1^* & & \vdots \\
\vdots & & \ddots & \ddots & & & & \vdots \\
\vdots & & & \ddots & \ddots & & & \vdots \\
\vdots & & 0^* & & \ddots & \ddots & & \vdots \\
0 & 0 & & & & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{array} \right)
\end{array} .
$$

Character of $L(\Lambda_0)$

The ground state path is $\mathfrak{p}_{\Lambda_0} = (\cdots \overline{1}\,1\,\overline{1}\,1\,\overline{1})$.
For $D = d = t = 2$, we obtain $u^{(0)} = -1$ and $u^{(1)} = 1$ and the corresponding partial order on odd colored integers:

$$\cdots \ll \begin{matrix} (-1)_{c_{\overline{1}}} \\ 1_{c_1} \end{matrix} \ll 1_{c_2} \ll \cdots \ll 1_{c_n} \ll 1_{c_{\overline{n}}} \ll \cdots \ll 1_{c_{\overline{2}}} \ll \begin{matrix} 1_{c_{\overline{1}}} \\ 3_{c_1} \end{matrix} \ll 3_{c_2} \ll \cdots$$

with the interlacing sequence

$$(2k+1)_{c_1} \ll (2k-1)_{c_{\overline{1}}} \ll (2k+1)_{c_1}.$$

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with the interlacing sequence

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The set $\mathcal{P}_2$ consists of the multi-grounded partitions into odd colored integers and grounded in $c_{\overline{1}}c_1$, and the generating function is given by

$$\frac{(-c_1 q, -c_{\overline{1}} q, \ldots, -c_n q, -c_{\overline{n}} q; q^2)_\infty}{(c_{\overline{1}} c_1 q^4; q^4)_\infty}.$$

Character of $L(\Lambda_1)$

The ground state path is $\mathfrak{p}_{\Lambda_0} = (\cdots \overline{1}\,1\,\overline{1}\,1\,\overline{1}\,1)$. For $D = d = t = 2$, we obtain $u^{(0)} = 1$ and $u^{(1)} = -1$, and the generating function of $\mathcal{P}_2$ is

$$\frac{(-c_1 q^3, -c_{\overline{1}} q^{-1}, -c_2 q, -c_{\overline{2}} q \ldots, -c_n q, -c_{\overline{n}} q; q^2)_\infty}{(c_{\overline{1}} c_1 q^4; q^4)_\infty}.$$

What we have done.

- We computed the character of standard level one modules of type $A_{n-1}^{(1)}(n \geq 2), B_n^{(1)}(n \geq 3), D_n^{(1)}(n \geq 4)$.
- We retrieved the character of standard level one modules of type $A_{2n}^{(2)}(n \geq 2), D_{n+1}^{(2)}(n \geq 3)$.
- We computed all the character of standard modules of type $A_1^{(1)}$ and derived partition identities involving absolute values.

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- We computed all the character of standard modules of type $A_1^{(1)}$ and derived partition identities involving absolute values.

- Compute the character of standard level one modules of type $C_n^{(1)}(n \geq 2)$.
- Compute the character of standard modules for all levels and all types.