Triangulations, Order Polytopes, and Generalized Snake Posets

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FPSAC
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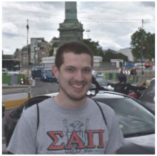

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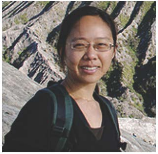

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Order Polytopes

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The order polytope is defined as

$$\mathcal{O}(P) = \left\{ \mathrm{x} = (x_1, \ldots, x_d) \in [0,1]^d : x_i \leq x_j \text{ for } i <_P j \right\}.$$

Order Polytopes: An Example

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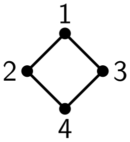
Let P be the diamond poset



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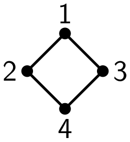
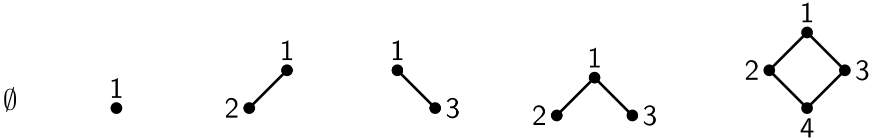


The six upper order ideals of P are



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Then $\mathcal{O}(P) = \{(x_1, x_2, x_3, x_4) \in [0,1]^4 : x_4 \leq x_2 \leq x_1 \text{ and } x_4 \leq x_3 \leq x_1\}$.

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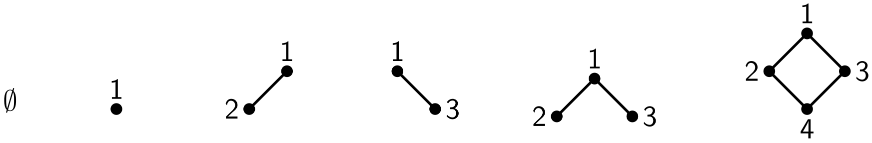
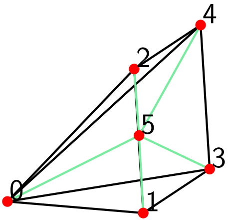
From the upper order ideals of P,



we get that $\mathcal{O}(P)$ is the convex hull of the points
$(0,0,0,0), (1,0,0,0), (1,1,0,0), (1,0,1,0)$, $(1,1,1,0)$ and $(1,1,1,1)$.



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- The vertices of $\mathcal{O}(P)$ correspond to the filters of P, i.e., the upper order ideals.
- Volume of $\mathcal{O}(P)$ is the number of linear extensions of P.

Generalized Snake Posets

Definition

For $n \in \mathbb{Z}_{\geq 0}$, a *generalized snake word* is a word of the form $\mathrm{w} = w_0 w_1 \cdots w_n$ where $w_0 = \varepsilon$ is the empty letter and w_i is in the alphabet $\{L, R\}$ for $i = 1, \ldots, n$. The *length* of the word is n, which is the number of letters in $\{L, R\}$.

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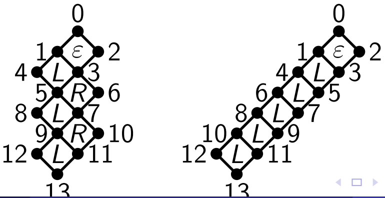
The snake poset $S_5 = P(\varepsilon LRLRL)$ and the ladder poset $\mathcal{L}_5 = P(\varepsilon LLLLL)$.



Volume of the Order Polytope of Generalized Snake Posets

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Theorem (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

For $n \geq 0$, let $\mathrm{w} = w_0 w_1 \cdots w_n$ be a generalized snake word. If $k \geq 0$ is the largest index such that $w_k \neq w_n$, then the normalized volume v_n of $\mathcal{O}(P(\mathrm{w}))$ is given recursively by

$$v_n = \mathrm{Cat}(n-k+1)v_k + (\mathrm{Cat}(n-k+2) - 2 \cdot \mathrm{Cat}(n-k+1))\, v_{k-1}$$

with $v_{-1} = 1$ and $v_0 = 2$.

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Corollary (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

The normalized volume of $\mathcal{O}(S_n)$ with $n \geq 0$ is given recursively by

$$v_n = 2v_{n-1} + v_{n-2},$$

with $v_{-1} = 1$ and $v_0 = 2$. These are the Pell numbers.

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Theorem (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

For any generalized snake word $\mathrm{w} = w_0 w_1 \cdots w_n$ of length n,

$$\mathrm{vol}\,\mathcal{O}(S_n) \leq \mathrm{vol}\,\mathcal{O}(P(\mathrm{w})) \leq \mathrm{vol}\,\mathcal{O}(\mathcal{L}_n).$$

Triangulations

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Consider a polytope $\mathcal{P} \subseteq \mathbb{R}^d$. A triangulation $\mathcal{T}$ of $\mathcal{P}$ is a subdivison of $\mathcal{P}$ into d-simplices.

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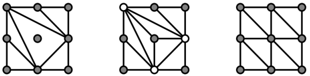

Figure: From "Existence of Unimodular Triangulations" by Haase et al.

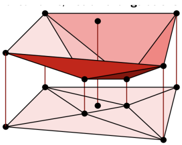

Figure: From "Triangulations" by De Loera et al.

Triangulations of Order Polytopes

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3. the simplex corresponding to a linear extension $(a_1, \ldots, a_d)$ of P is

$$\sigma_{a_1,\ldots,a_d} = \left\{\mathrm{x} \in [0,1]^d : x_{a_1} \leq x_{a_2} \leq \cdots \leq x_{a_d}\right\},$$

with vertex set $\{0, \mathrm{e}_{a_d}, \mathrm{e}_{a_{d-1}} + \mathrm{e}_{a_d}, \ldots, \mathrm{e}_{a_1} + \cdots + \mathrm{e}_{a_d} = 1\}$.

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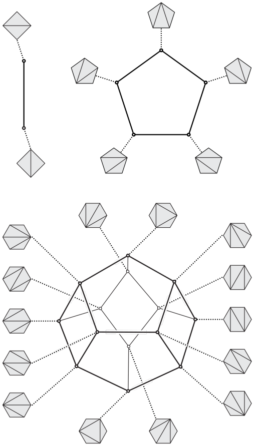

The Order Polytope of Meet-irreducibles

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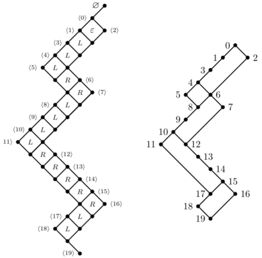

Figure: The lattice $\hat{P}(\mathrm{w})$ for $\mathrm{w} = \varepsilon L^3 R^2 L^4 R^5 L^2$ (left) and its poset of meet-irreducibles $Q_{\mathrm{w}} = \mathrm{Irr}_{\wedge}(\hat{P})$.

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Theorem (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

For $\mathrm{w} \in \mathcal{V}$*, every vertex of the secondary polytope of* $\mathcal{O}(Q_{\mathrm{w}})$ *is a unimodular triangulation. Thus, every triangulation of* $\mathcal{O}(Q_{\mathrm{w}})$ *is unimodular.*

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Theorem (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

Let $\mathrm{w} \in \mathcal{V}$ *have length* k*. The canonical triangulation of* $\mathcal{O}(Q_{\mathrm{w}})$ *admits exactly* $k+1$ *flips.*

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Theorem (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

Let $\mathrm{w} = \varepsilon L^{n-1}$, *and* $Q_{\mathrm{w}} = \mathrm{Irr}_{\wedge}(\hat{P}(\mathrm{w}))$. *The flip graph of triangulations of* $\mathcal{O}(Q_{\mathrm{w}})$ *is the Cayley graph of the symmetric group* $\mathfrak{S}_{n+1}$ *with the simple transpositions as the generating set.*

Twists

Twists

Definition

Given a ladder $\mathcal{L}^i$, define $\tau_i \in \mathfrak{S}_{|V_0|}$ to be the permutation of V_0 such that for $v \in V_0$,

$$\tau_i(v) = \begin{cases} x_{j-1}, & \text{if } v = x_j \text{ and } j \in [s] \text{ is even,} \\ x_{j+1}, & \text{if } v = x_j \text{ and } j \in [s] \text{ is odd,} \\ v, & \text{otherwise.} \end{cases}$$

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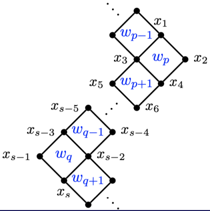

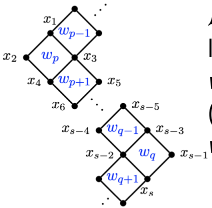

$\mathcal{L}^i$ in $\widehat{P}$ containing boxes with labels $w_p, \ldots, w_q$, where $w_p < w_{p+1} < \cdots < w_q$. The left (right) represents the case where $w_q = L$ ($w_q = R$).

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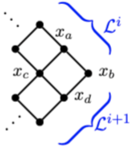

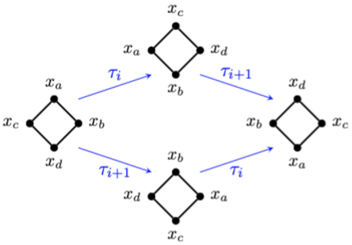

Definition

Let $\mathfrak{T}(\mathrm{w})$ denote the subgroup of $\mathfrak{S}_{|V_0|}$ generated by the set of the τ_i's. We call $\mathfrak{T}(\mathrm{w})$ *the twist group of* $\widehat{P}(\mathrm{w})$. Elements of $\mathfrak{T}(\mathrm{w})$ are called *twists* and the elements τ_i are called *elementary twists*.

Twists

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Theorem (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

Let $\mathrm{w} \in \mathcal{V}$, $Q_{\mathrm{w}} = \mathrm{Irr}_{\wedge}(\widehat{P}(\mathrm{w}))$, *and let* $\mathcal{T}$ *and* $\tau(\mathcal{T})$ *be two triangulations of* $\mathcal{O}(Q_{\mathrm{w}})$ *where* τ *is a twist. If* $\mathcal{T} = \mathcal{T}_Z^+$ *can be flipped at circuit* Z *and* $\tau(\mathcal{T}_Z^+) = \tau(\mathcal{T}_Z^+)_{\tau(Z)}^+$, *then* $\tau(\mathcal{T}_Z^+)_{\tau(Z)}^- = \tau(\mathcal{T}_Z^-)$. *In other words, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{T}_Z^+ & \xrightarrow{\textit{flip in } Z} & \mathcal{T}_Z^- \\ \Big\downarrow \textit{twist} & & \Big\downarrow \textit{twist} \\ \tau(T_Z^+) = \tau(\mathcal{T}_Z^+)_{\tau(Z)}^+ & \xrightarrow{\textit{flip in } \tau(Z)} & \tau(\mathcal{T}_Z^+)_{\tau(Z)}^- = \tau(\mathcal{T}_Z^-) \end{array}$$

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Corollary (MB-BB-DH-KS-JV-ARVM-MY, 2022+)

Let $\mathrm{w} \in \mathcal{V}$, $Q_{\mathrm{w}} = \mathrm{Irr}_{\wedge}(\widehat{P}(\mathrm{w}))$, *and* $\mathcal{T}$ *&* $\tau(\mathcal{T})$ *be two triangulations of* $\mathcal{O}(Q_{\mathrm{w}})$. *Then* $\mathcal{T}$ *and* $\tau(\mathcal{T})$ *admit the same number of flips.*

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Let $\mathrm{w} \in \mathcal{V}$ *and* $Q_{\mathrm{w}} = \mathrm{Irr}_{\wedge}(\widehat{P}(\mathrm{w}))$. *The canonical triangulation* $\mathcal{T}_{\mathrm{w}}$ *of* $\mathcal{O}(Q_{\mathrm{w}})$ *is a regular triangulation, and for any twist* τ, $\tau(\mathcal{T}_{\mathrm{w}})$ *is also a regular triangulation.*

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Let $\mathrm{w} \in \mathcal{V}$ *and* $Q_{\mathrm{w}} = \mathrm{Irr}_{\wedge}(\widehat{P}(\mathrm{w}))$. *The canonical triangulation* $\mathcal{T}_{\mathrm{w}}$ *of* $\mathcal{O}(Q_{\mathrm{w}})$ *is a regular triangulation, and for any twist* τ, $\tau(\mathcal{T}_{\mathrm{w}})$ *is also a regular triangulation.*

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- The twist group $\mathfrak{T}(\mathrm{w})$ acts on the canonical triangulation of $\mathcal{O}(Q_{\mathrm{w}})$.
- Any twist τ, $\tau(\mathcal{T}_{\mathrm{w}})$ corresponds to a framed triangulation of $\mathcal{F}_{G_{Q_{\mathrm{w}}}}$, by DKK we know are regular.

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The End

¡Gracias!