Rigidity with few locations: Vertex Spanning Planar Laman Graphs in Triangulated Surfaces

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The Hebrew University,
based on joint work with **Simion Tarabykin**

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What is infinitesimal rigidity?

$G = ([n], E)$ is a finite graph. ($[n] = \{1, 2, \ldots, n\}$.)
$p : [n] \to \mathbb{R}^d$.
Assume $n \geq d + 1$.

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$$\mathrm{rank(R(G,p)) = dn - \binom{d+1}{2}},$$

where the *rigidity matrix* $R(G, p) \in M_{dn \times |E|}(\mathbb{R})$ is defined by:
column $ij \in E$ equals (say $i < j$)
$[0, \ldots, 0, p(i) - p(j), 0 \ldots, 0, p(j) - p(i), 0, \ldots, 0]^T$.

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Fact: for generic p, $\mathrm{rank}(\mathrm{R}(\mathrm{K_n}, \mathrm{p})) = \mathrm{dn} - \binom{\mathrm{d}+1}{2}$.

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G is (generically) *d-rigid* if (G, p) is infinitesimally rigid for some (equivalently all) generic $p : [n] \to \mathbb{R}^d$.

Examples:

① $G = \Delta$, $\mathbb{R}^2$:

p = flat triangle: 

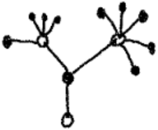
Non trival infi. motion

$\Rightarrow (G,p)$ is not infi. rigid.

p generic:  is infi. rigid.

2) $G =$  $\Rightarrow$ G is 2-rigid.

G is NOT 3-rigid $\left(|E| < 3 \cdot 4 - \binom{3+1}{2} = 6\right)$

③ $G =$ 

$p: V \to \mathbb{R}^1$

$\bullet \mapsto 1$

$\circ \mapsto -1$

(G,p) is infi. rigid.

Main objective

$G = (V, E)$ is a finite graph.
$A \subseteq \mathbb{R}^d$.

Definition

G is *A-rigid* if there exists $p : V \to A$ such that the framework (G, p) is infinitesimally rigid.

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$\mathcal{F}$ a family of generically d-rigid graphs. $\mathcal{F}(n)$ is its subfamily of $G \in \mathcal{F}$, $|V(G)| \leq n$.

Definition

$\mathcal{F}$ is *d-rigid with c locations* if there exists $A \subseteq \mathbb{R}^d$ of size c s.t all $G \in \mathcal{F}$ are A-rigid.
$c_d(\mathcal{F})$ is the minimal such c.

Main interest:

1. Families with bounded $c_d(\mathcal{F})$.
2. Growth of $c_d(\mathcal{F}(n))$ as $n \to \infty$.

All d-rigid graphs

$\mathcal{F}_d = \{$all d-rigid graphs$\}$.

Fekete-Jordan 2005

$c_1(\mathcal{F}_1) = 2$. (As a spanning tree is bipartite.)
$c_2(\mathcal{F}_2(n)) = \Omega(\sqrt{n})$.

Their argument shows: for $d \geq 2$, $c_d(\mathcal{F}_d(n)) = \Omega(\sqrt{n})$.
Sketch: let H be minimally d-rigid on $k \geq d$ vertices.
$G = G(H)$ is obtained by: for each pair of vertices $v, u \in H$ choose a d-subset $B = B(v, u) \subseteq V(H)$ containing them and connect a new vertex v_B to all vertices in B.
Then $|V(G)| = k + \binom{k}{2}$, and in a d-rigid realization of G each vertex of H must have a different location!

Király 2021

$c_2(\mathcal{F}_2(n)) = \Theta(\sqrt{n})$.

Planar graphs

Király 2021 (answers Whiteley)

$c_2(\text{Planar Laman}) \leq 26.$ (Laman:= edge-minimal 2-rigid)

Adiprasito-N. 2020

$c_3(\text{Maximal Planar}) \leq 76.$

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The main algebraic statement, allowing inductive proofs in both results, is about moving vertices into A:

Adiprasito-N. 2020, also Király 2021

Assume (G, p) is infinitesimally rigid in $\mathbb{R}^d$, $\deg_G(v) = c$, $A \subseteq \mathbb{R}^d$ with generic coordinates, $|A| \geq \binom{d+c}{d}$.
Then there exists $a \in A$ s.t. (G, p') is infinitesimally rigid, where $p' : V \to \mathbb{R}^d$ is defined by $p'(v) = a$ and $p'(u) = p(u)$ for all $u \in V - v$.

Graphs on surfaces

Let M_g denotes the surface of genus g (orientable or not).
Let $\mathcal{F}(M_g)$ be the family of graphs of triangulations of M_g.

Fogelsanger 1988

$c_3(\mathcal{F}(M_g)) \leq \aleph_0$ for all g,
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$c_3(\mathcal{F}(M_g)) \leq c(g)$ for some constant $c(g)$ depending on g.

Let $\mathcal{L}(M_g)$ be the Laman graphs embedable in M_g.

Király 2021

$c_2(\mathcal{L}(M_g)) \leq c(g)$ for some constant $c(g) = O(\sqrt{g})$.

Open: are there absolute constants c_3, c_2 s.t. for all g:
$c_3(\mathcal{F}(M_g)) \leq c_3$?
$c_2(\mathcal{L}(M_g)) \leq c_2$?

New results

Intermediate problem

Does every graph of a triangulation of a surface M_g contain a vertex spanning planar Laman subgraph?

Note: if YES then $c_2(\mathcal{F}(M_g)) \leq 26$.

Theorem (N.- Simion Tarabykin)

YES if the Euler characteristic $\chi(M_g) \geq 0$.

More strongly

N.-Tarabykin

- Every triangulation of the projective plane $\mathbb{R}P^2$ contains a spanning disc.
- Every triangulation of the Torus T contains a spanning cylinder.
- Every triangulation of the Klein bottle K contains a vertex spanning, planar, 2-dimensional complex; it is either a cylinder, or a pinched disc, or a connected sum of two triangulated discs along a triangle.

Then these vertex spanning subcomplexes are indeed 2-rigid, hence contain a spanning planar Laman subgraph, and Király's result apply.

Irreducible triangulations

A triangulation Δ of M_g is irreducible if each contraction of an edge of Δ changes the tolopogy; equivalently, each edge belongs to an *empty* triangle of Δ.

Barnette-Edelson 1988/9

For all g, M_g has finitely many minimal triangulations.

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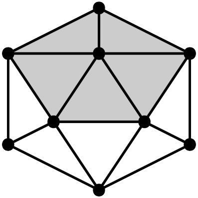
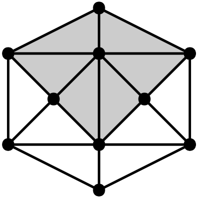
When $\chi(M_g) \geq 0$ the minimal triangulations are characterized:
2 such $\mathbb{R}P^2$: Barnette 1982.
21 such T: Lavrenchenko 1990.
29 such K: Lavrenchenko-Negami 1997, Sulanke 2006.

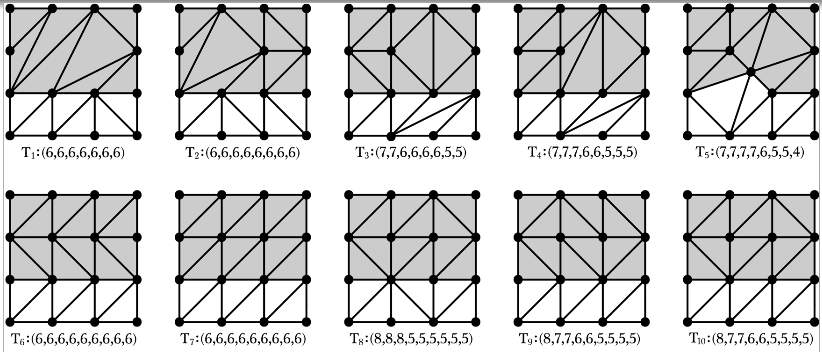


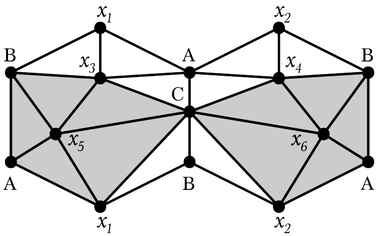
$\mathrm{RP2}_1$:(5,5,5,5,5,5)



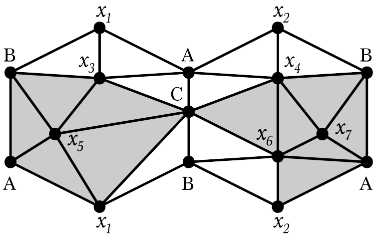
$\mathrm{RP2}_2$:(6,6,6,6,4,4,4)

Some spanning subcomplexes in minimal triangulations





Kc_1:(8,8,8,5,5,5,5,5,5)



Kc_2:(9,9,7,6,6,5,5,5,4,4)

Induction via vertex splits

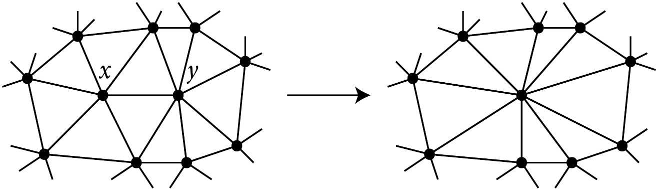
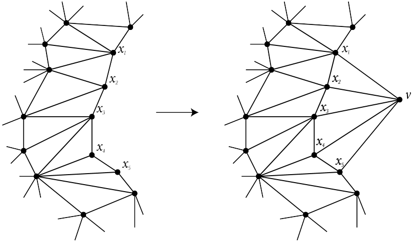


Figure: Above: edge contraction; reverse arrow for vertex split. Below: cone over boundary interval in the spanning subsurface.



We want the vertex split $\Delta \to \Delta'$ to allow an extension $S' \subseteq \Delta'$ of the spanning disc/cylinder/etc $S \subseteq \Delta$.

Extension

Definition: extension

A spanning subsurface $S \subseteq \Delta$ is *extendible* if for every vertex split $\Delta \to \Delta'$ there exists $S' \subseteq \Delta'$ s.t. either
(i) S' is obtained from S by a split at the same vertex, or
(ii) S' is obtained from S by coning over an interval in its boundary.

Note: then $S' \subseteq \Delta'$ is spanning and homeomorphic to S.

Theorem (N.-Tarabykin)

Let Δ triangulate some M_g, and let $S \subseteq \Delta$ be a vertex spanning subsurface. Then:
(1) S is extendible in Δ iff it includes at least one edge from every triangle in Δ.
(2) If S is extendible then it has an extendible extension $S' \subseteq \Delta'$.

The easy direction

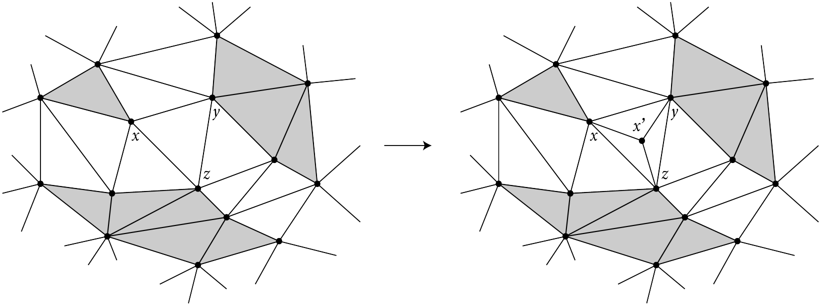


Figure: Non extendible subcomplex.

Note: all S we chose in irreducible triangulations are extendible subsurfaces, except for the 4 in the "crosscap" triangulations of the Klein bottle, which are *pinched* discs.

How to choose an extendible S'?

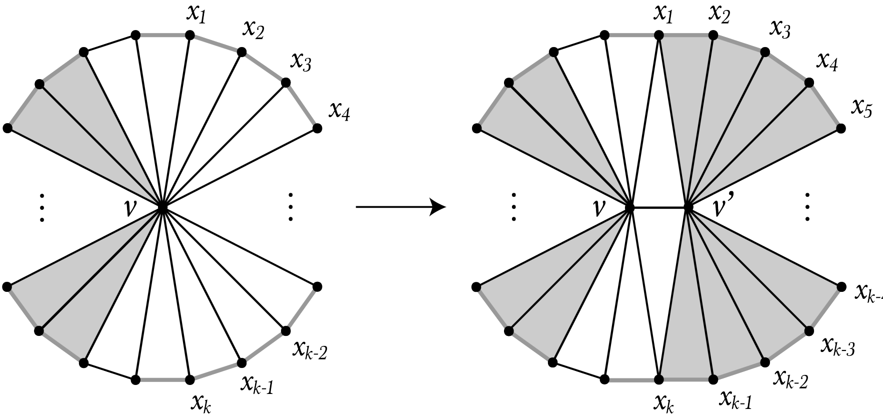


Figure: $S \to S'$ via coning over boundary interval.

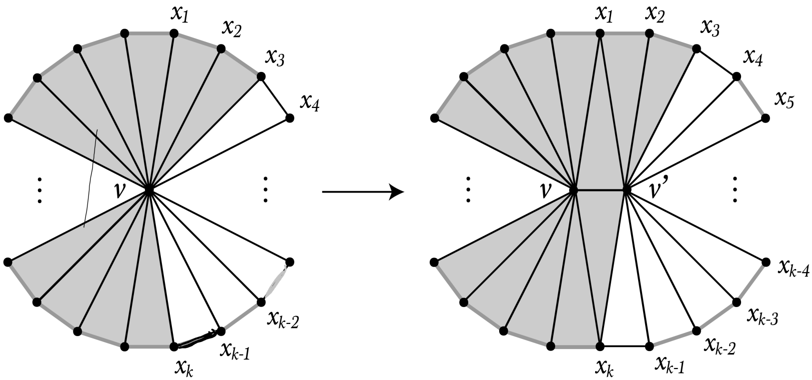
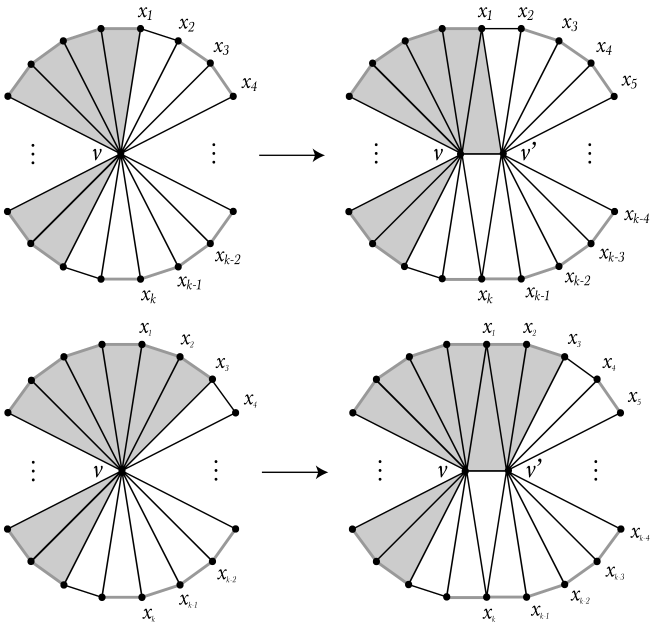
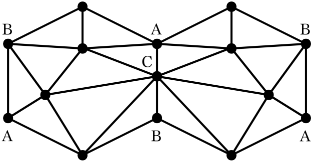


Figure: $S \to S'$ via vertex split.

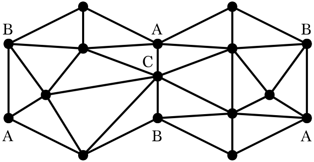

x1
x2
x3
x4
v
xk-2
xk-1
xk
x1
x2
x3
x4
x5
v
v'
xk-4
xk-3
xk-2
xk-1
xk
x1
x2
x3
x4
v
xk-2
xk-1
xk
x1
x2
x3
x4
x5
v
v'
xk-4
xk-3
xk-2
xk-1
xk

What's left: crosscap triangulation of K

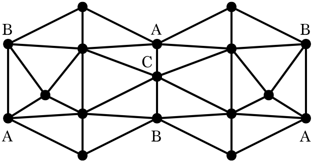
ABC is a noncontractible cycle in each of the 4 crosscap irreducible triangulations:



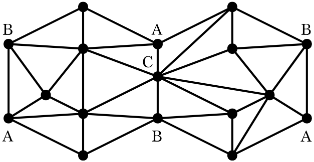
Kc_1:(8,8,8,5,5,5,5,5,5)



Kc_2:(9,9,7,6,6,5,5,5,4,4)



Kc_3:(10,10,6,6,6,6,6,4,4,4,4)

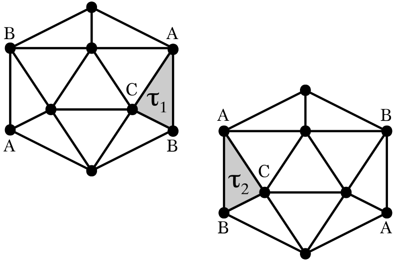


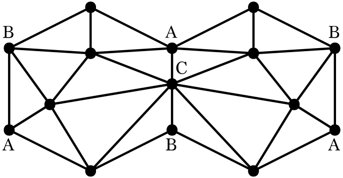
Kc_4:(10,8,8,6,6,6,6,4,4,4,4)

Each is a connected sum of two $\mathbb{R}P^2$'s along the triangle ABC; this triangle can be part of the spanning disc in each $\mathbb{R}P^2$.

Does ABC survive the vertex splits?

If YES, then again the triangulation Δ of K is a connected sum of two $\mathbb{R}P^2$'s along ABC, and ABC can be taken as a triangle in each spanning disc.
The connected sum of those discs is a planar strongly connected 2-complex.





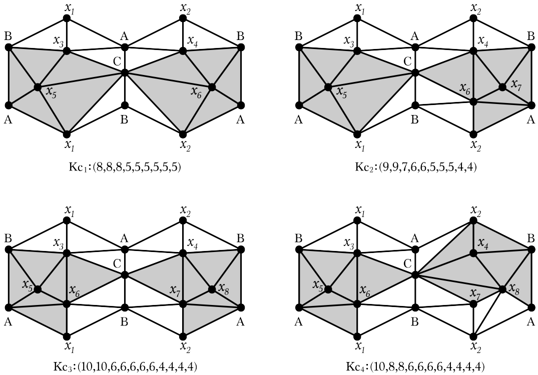
If NOT

Then some vertex split induced a vertex split of the cycle ABC.

Commutativity claim

The vertex splits can be rearranged s.t. the first one splits ABC.

If C splits first (similaly for A, B), choose S a spanning pinched disc at C.



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Kc_4:(10,8,8,6,6,6,6,4,4,4,4)

From a pinched disc to a cylinder

The first split makes S' a spanning cylinder.

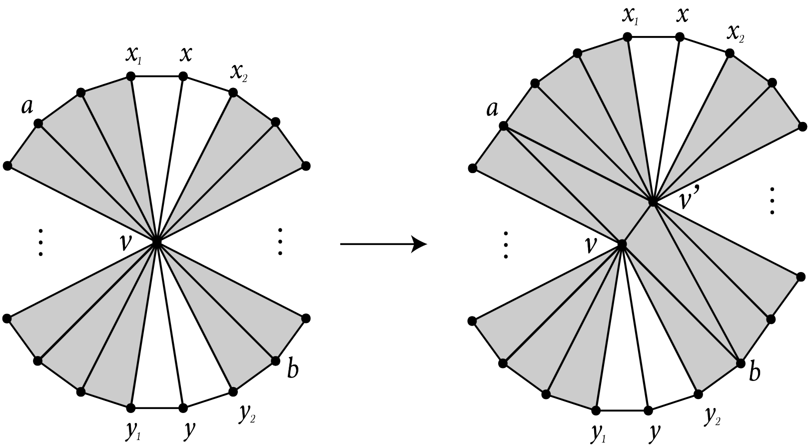

Figure: Resolving a singularity.

The Extension Theorem shows that further splits preserve having a spanning cylinder. This completes the proof. □

Open problems

- All surfaces: (i) Is $c_3(\cup_g \mathcal{F}(M_g))$ finite?
  (ii) Is $c_2(\cup_g \mathcal{L}(M_g))$ finite?
  (iii) If NO in (i), is $c_2(\cup_g \mathcal{F}(M_g))$ finite? E.g. via:

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THANK YOU!

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